# ON THE SOME PARTICULAR SETS

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**Abstract**

For $t$ an integer, a $P_t$ set is defined as a set of $m$ positive integers with the property that the product of its any two distinct element increased by $t$ is a perfect square integer.

In this study, the certain special $P_{-5}$, $P_{+5}$, $P_{-7}$ and $P_{+7}$ sets with size three are considered. It is demonstrated that they cannot be extended to $P_{-5}$, $P_{+5}$, $P_{-7}$ and $P_{+7}$ with size four. Also, some properties of them are proved.

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# BAZI ÖZEL KÜMELER ÜZERİNE

**Özet**

Bir $t$ tamsayısı için $P_t$ kümesi, herhangi iki tane farklı elemanının çarpımının $t$ fazlası bir tamkare olma özelliğine sahip $m$ tane pozitif tamsayıdan oluşan bir küme olarak tanımlanır.

Bu çalışmada, üç elemanlı bazı $P_{-5}$, $P_{+5}$, $P_{-7}$ ve $P_{+7}$ kümeleri gözönüne alınıyor. Bu kümelerin dört elemanlı $P_{-5}$, $P_{+5}$, $P_{-7}$ ve $P_{+7}$ kümelerine genişletilemez olduğu gösteriliyor. Ayrıca, bu kümelerin bazı özellikleri kanıtlanıyor.

**Anahtar Kelimeler:** $P_t$ kümeleri, Kongrüanslar, Karşıtlık.

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## 1. INTRODUCTION

Let $t$ be an integer. A $\mathrm{P_t}$-set of size m is a set $B = \{\mathrm{x_1}, \mathrm{x_2}, \mathrm{x_3} \ldots . \mathrm{x_m}\}$ of distinct positive integers for which $\mathrm{x_i x_j} + \mathrm{t}$ is the square of an integer whenever $\mathrm{i} \neq \mathrm{j}$. If there exists a positive integer $\mathrm{n} \notin B$ such that $B \cup \{\mathrm{n}\}$ is still a $\mathrm{P_t}$-set, then the $\mathrm{P_t}$-set $B$ can be extended.

The simultaneous Pell equations have been studied by most of authors like Anglin, Baker, Dickson, Mordell, Davenport, Cohn, Mohanty, Ramasamy, Pinch, Ponnudurai, Tzanakis, etc…In this topic, many authors applied Baker-Davenport method [2] provided set {1, 3, 8,120} of size four to investigate similar problems. Besides, some authors such as Kanagasabapathy and Ponnudurai [8], Brown [3] studied on the number of the solutions of simultaneous Pell equations. The other like Mohanty and Ramasamy [12], Gopalan [6] as well as Filipin, Fujita and Mignotte [5] introduced the concept of characteristic number of two simultaneous Pell's equations.

Moreover, Anglin [1] presented a method for solving a system of Pell's equations with the parameters in the boundry. Tzanakis [15] provided elliptic logarithm method using linear forms in elliptic logarithms. In [9], Katayama also partially described elliptic logarithm method for simultaneous Pell equations. Also, readers can look into [4, 7, 10, 11, 13, 14] references for more information about the $P_t$ sets and Pell equations.

In this research paper, we will prove the sets $P_{-5} = \{1,6,9\}$ , $P_{-5} = \{1,9,14\}, P_{+5} = \{1,4,11\}$, $P_{-7} = \{1,16,23\}$, $P_{-7} = \{1,16,176\}, P_{-7} = \{2,8,16\}$ and $P_{+7} = \{1,9,18\}$ can not be extended with size four $P_{-5}$, $P_{+5}$, $P_{-7}$ and $P_{+7}$ sets. Also, we will demonstrate some properties of such sets.

## 2. PRELIMINARIES

**Definition 2.1. ([14])** If $m \in N$ and $a \in Z$ with $\gcd(a, m) = 1$, then a is said to be a quadratic residue modulo $m$ if there exists an integer $x$ such that

$$x^2 \equiv a \ (mod\ m) \tag{2.1}$$

and if equivalence has no such solution, then a is a quadratic nonresidue modulo n.

**Definition 2.2**. **([14])** If $a \in Z$ and $p > 2$ is prime,then

$$\left(\frac{a}{p}\right) = \begin{cases} 0\,, & \text{if p|a} \\ 1\,, & \text{if a is quadratic residue modp} \\ -1\,, & \text{otherwise} \end{cases} \tag{2.2}$$

and $\left(\frac{a}{p}\right)$ is called the Legendre Symbol of $a$ with respect to $p$.

The following is a fundamental result on quadratic residuacity modulo n. This term means the determination of whether n integer to be a quadratic residue or a non-residue modulo n.

**Theorem 2.1. ([14])** If $p \neq q$ are odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \tag{2.3}$$

where $\left(\frac{\cdot}{\cdot}\right)$ represents Legendre symbol.

**Theorem 2.2. ([14])** For any odd prime p,

$$\left(\frac{2}{p}\right) \equiv (-1)^{(p^2-1)/8}\ (mod\ p) \tag{2.4}$$

**Theorem 2.3. ([14])** Let $s > 1$ be an integer, $c \in Z$ with $\gcd(c, s) = 1$ and

$$s = 2^{a_0} \prod_{j=1}^{m} {p_j}^{a_j} \tag{2.5}$$

be the canonical prime factorization of $s$ where $a_0 \geq 0$ and $a_j \in N$ for the distinct odd primes $p_j$ , $j = 1,2, \dots, m$. Then

$$x^2 \equiv c\ \ (mod\ n) \tag{2.6}$$

is solvable if and only if

$$\left(a/p_j\right) = 1 \tag{2.7}$$

for all $j = 1,2, \dots, m$ and $a \equiv 1\ \ (mod\ \ \gcd(8,2^{a_0}))$.

**Theorem 2.4. ([14])** If $m, n \in$ N are odd and relatively primes, then

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2}\cdot\frac{n-1}{2}} \tag{2.8}$$

holds.

## 3. MAIN THEOREMS AND RESULTS

**Theorem 3.1.** The set $P_{-5} = \{1,6,9\}$ can not be extended to the set $P_{-5}$ with size 4.

**Proof.** We assume that $P_{-5} = \{1,6,9\}$ can be extended for any positive integer $d$.

i.e, $\{1, 6,9, d\}$ is a $P_{-5}$ set. We can find $x, y, z$ integers such that;

$$d - 5 = x^2 \tag{3.1}$$

$$6d - 5 = y^2 \tag{3.2}$$

$$9d - 5 = z^2 \tag{3.3}$$

by dropping d from (3.1) and (3.3) we obtain

$$z^2 - 9x^2 = 40 \tag{3.4}$$

In (3.4), we can write left side as difference of two squares since 9 is a perfect square $(z - 3xz+3x=40$. Also, it is clear that 40 can be factorized as finitely. So, integer solutions of (3.4) are obtained as following:

$$(x, z) = (\pm 3, \pm 11) \tag{3.5}$$

or

$$(x, z) = (\pm 1, \pm 7) \tag{3.6}$$

Eliminating $d$ from (3.1) and (3.2) simultaneously, then we obtain

$$y^2 - 6x^2 = 25 \tag{3.7}$$

Using the solutions of equation (3.5) and substituting $x^2 = 9$ into (3.7) we have $y^2 = 79$ which $y$ is not an integer solution.

In a similar way, substituting (3.6) solutions ($x^2 = 1$) into the (3.7), we get $y^2 = 31$. This shows that $y$ is not integer for the solution of (3.7). Thus, there is no a such $d \in Z$ and the set $P_{-5} = \{1,6,9\}$ can not be extended.

**Theorem 3.2.** The set $P_{-5} = \{1,9,14\}$ is nonextendible.

**Proof.** It can be proved in a similar way of the proof of Theorem 3.1. Suppose that $P_{-5} = \{1,9,14\}$ can be extended for any positive integer $d$. It means that $\{1, 9, 14, \mathrm{d}\}$ is a $P_{-5}$ set.

We can find $x, y, z$ integers such that;

$$d - 5 = x^2 \tag{3.8}$$

$$9d - 5 = y^2 \tag{3.9}$$

$$14d - 5 = z^2 \tag{3.10}$$

by dropping d from (3.8) and (3.9) we obtain $y^2 - 9x^2 = 40$ which correspond to (3.4) equation. In the previous proof we solved this equation and obtained solutions as $(y, x) = (\pm 11, \pm 3)$ or $(y, x) = (\pm 7, \pm 1)$. Eliminating $d$ from (3.8) and (3.10) simultaneously, then we get

$$z^2 - 14x^2 = 65 \tag{3.11}$$

Using $(y, x) = (\pm 11, \pm 3)$ solution and substituting $x^2 = 9$ into the (3.11), we obtain $z^2 = 191$ which $z$ is not an integer.

In a similar way, considering $(y, x) = (\pm 7, \pm 1)$ solutions and substituting $x^2 = 1$ into the (3.11), we get $z^2 = 79$. This shows that $z$ is not integer.

Thus, there is no such $d \in Z$ and the set $P_{-5} = \{1,9,14\}$ can not be extended.

**Theorem 3.3.** The set $P_{+5} = \{1,4,11\}$ is nonextendable to set $P_{+5}$ with size four.

**Proof.** We assume that the set $\{1,4,11,d\}$ is a $P_{+5}$ for any positive integer $d$. If we consider the definition of $P_{+5}$, then we have

$$d + 5 = x^2 \tag{3.12}$$

$$4d + 5 = y^2 \tag{3.13}$$

$$11d + 5 = z^2 \tag{3.14}$$

We have to find integers $x, y, z$, satisfying (3.12), (3.13) and (3.14). From (3.12) and (3.13) we get

$$4x^2 - y^2 = 15 \tag{3.15}$$

and from (3.12) and (3.14) we have

$$11x^2 - z^2 = 50 \tag{3.16}$$

By the same manner of the proof of above theorems and factorising (3.15) we get;

$$(2x - y)(2x + y) = 15 \tag{3.17}$$

If we get the solutions of equation (3.17), we obtain (x,y)=( $\pm 4, \pm 7$) and (x,y)=($\pm 2, \pm 1$). If we substituting $x^2 = 16$ or $x^2 = 4$ into the (3.16) then we obtain $z^2 = 126$ which $z$ is not an integer or $z^2 = -6$ which is impossible, consecutively. So, there is no any integer $z$ satisfying the equation (3.16).

Hence, the set $P_{+5} = \{1,4,11\}$ is non-extendable.

**Theorem 3.4.** The set $P_{+7} = \{1,9,18\}$ can not extendible.

**Proof.** Suppose that the set $\{1,9,18, d\}$ is a $P_{+7}$ for any positive integer $d$. Using the definition of set $P_{+7}$, then we obtain

$$d + 7 = x^2 \tag{3.18}$$

$$9d + 7 = y^2 \tag{3.19}$$

$$18d + 7 = z^2 \tag{3.20}$$

We have to find integers $x, y, z$, holding (3.18), (3.19) and (3.20). From (3.18) and (3.19) we get

$$9x^2 - y^2 = 56 \tag{3.21}$$

and by using (3.19) and (3.20) we obtain

$$-z^2 + 2y^2 = 7 \tag{3.22}$$

By factorising (3.21) we have

$$(3x - y)(3x + y) = 56 \tag{3.23}$$

If we search the solutions of the (3.23), we get $(x, y) = (\pm 5, \pm 13)$ and $(x, y) = (\pm 3, \pm 5)$. If we substituting $y^2 = 169$ or $y^2 = 25$ into the (3.22), then we get $z^2 = 331$ or $z^2 = 43$ not integer solution of (3.22) consecutively.

Therefore, the set $P_{+7} = \{1,9,18\}$ is nonextendable.

**Theorem 3.5.** The set $P_{-7} = \{2,8,16\}$ is nonextendable.

**Proof.** We assume that the set $\{2,8,16, d\}$ is a $P_{-7}$ for any positive integer $d$. By considering the definition of the set $P_{-7}$, then we get

$$2d - 7 = x^2 \tag{3.24}$$

$$8d - 7 = y^2 \tag{3.25}$$

$$16d - 7 = z^2 \tag{3.26}$$

We have to find integers $x, y, z,$ satisfying above equations. From (3.24) and (3.25), we get

$$y^2 - 4x^2 = 21 \tag{3.27}$$

and by using (3.25) and (3.26), we obtain

$$z^2 - 2y^2 = 7 \tag{3.28}$$

Considering the factorization of (3.27), we have

$$(y - 2x)(y + 2x) = 21 \tag{3.29}$$

We obtain (x,y)=( $\pm 5$, $\pm 11$) or (x,y)=( $\pm 1$, $\pm 5$). If we substituting $y^2 = 121$ or $y^2 = 25$ into the (3.28), then we get $z^2 = 249$ or $z^2 = 57$ consecutively. So, there is no any integer $z$ holding the (3.28) equation.

Hence, the set $P_{-7} = \{2,8,16\}$ is non-extendable.

**Theorem 3.6.** The sets $P_{-7} = \{1,16,23\}$ and $P_{-7} = \{1,16,176\}$ are nonextendable.

**Proof.** Let the set $\{1,16,23, d\}$ is a $P_{-7}$ for any positive integer $d$. Then we have

$$d - 7 = x^2 \tag{3.30}$$

$$16d - 7 = y^2 \tag{3.31}$$

$$23d - 7 = z^2 \tag{3.32}$$

From (3.30) and (3.31) we obtain

$$y^2 - 16x^2 = 105 \tag{3.33}$$

and by using (3.30) and (3.32) we obtain

$$z^2 - 23x^2 = 154 \tag{3.34}$$

Using the factorization of (3.33) we get

$$(y - 4x)(y + 4x) = 105 \tag{3.35}$$

We get $(x, y) = (\pm 13, \pm 53)$, $(x, y) = (\pm 4, \pm 19)$, $(x, y) = (\pm 2, \pm 13)$ or $(x, y) = (\pm 1, \pm 11)$. If we substituting $x^2 = 169$ into the (3.34), then we get $z^2 = 4041$ which $z$ isn't an integer holding the (3.34) equation. In the same manner, substituting $x^2 = 16$ , $x^2 = 4$ or $x^2 = 1$ into the (3.34), we get $z^2 = 522$, $z^2 = 246$ or $z^2 = 177$ that $z$ is not an integer holding the (3.34) equation either.

Therefore, the set $P_{-7} = \{1,16,23\}$ is non-extendable.

For the set $P_{-7} = \{1,16,176\}$, we have (3.30) and (3.31) with the equation

$$176d - 7 = z^2 \tag{3. 36}$$

From (3.30) and (3.36), we have

$$z^2 - 176x^2 = 1225 \tag{3.37}$$

If we put the solutions of (3.33) into the (3.37), then we get $z^2 = 30969$, $z^2 = 4041$, $z^2 = 1929$ or $z^2 = 1401$ which $z$ isn't an integer holding the (3.37). As a consequence, $P_{-7} = \{1,16,176\}$ can not extendible.

**Theorem 3.7.** There is no set $P_{-5}$ includes any multiple of 4, 11 or 17.

**Proof. (i)** Suppose that $m$ is an element of set $P_{-5}$. If $4r$ is also an element of set $P_{-5}$ for $r \in Z$, then

$$4rm - 5 = a^2 \tag{3.38}$$

has to satisfy for integer $a$. If we apply (modulo 4) into the (3.38), we have

$$a^2 \equiv 3 \ (mod\ 4) \tag{3.39}$$

If $a$ is even integer, then $a^2 \equiv 0 \ (mod\ 4)$ holds. If $a$ is odd integer, then $a^2 \equiv 1 \ (mod\ 4)$ holds. So, there is no an integer satisfying $a^2 \equiv 3 \ (mod\ 4)$.

Hence, there is no set $P_{-5}$ includes any multiple of 4.

**(ii)** Assume that $m$ is an element of set $P_{-5}$. If $11s$ is also an element of set $P_{-5}$ for $s \in Z$ ,then

$$11sm - 5 = b^2 \tag{3.40}$$

has to satisfy for integer $b$. If we apply (modulo 11) on the (3.40), we get

$$b^2 \equiv 6 \ (mod\ 11) \tag{3.41}$$

Using Theorem 2.2. and the Definition 2.2 or considering residue classes (modulo 11), we obtain $b^2 \equiv 1,3,4,5,9 \ (mod\ 11)$ which not satisfies $b^2 \equiv 6 \ (mod\ 11)$. (It means that there is no any $b$ integer holding (3.41)). This is a contradiction. So, there is no $P_{-5}$ set contains any multiple of 11.

**(iii)** Similarly, if we suppose that $m$ is an element of set $P_{-5}$ and $17k$ is also an element of set $P_{-5}$ for $k \in Z$, then we have,

$$c^2 \equiv 12 \ (mod\ 17) \tag{3.42}$$

Using residue classes (modulo 17), we have $c^2 \equiv 1,2,4,8,9,13,15 \ (mod\ 17)$ which implies that there is no integer holding $c^2 \equiv 12 \ (mod\ 17)$. This is a contradiction. As a consequence, there is no set $P_{-5}$ involves any multiple of 17.

**Theorem 3.8.** There is no set $P_{+5}$ contains any multiple of 3,7 or 13.

**Proof**. **(i)**Assume that $n$ is an element of set $P_{+5}$. If $3u$ is also an element of set $P_{+5}$ for $u \in Z$ ,then

$$3un + 5 = A^2 \tag{3.43}$$

has to satisfy for some integer $A$. If we apply (modulo 3) on the (3.43), we obtain

$$A^2 \equiv 2 \ (mod\ 3) \tag{3.44}$$

By using Theorem 2.2 and the Definition 2.2, we have

$$\left(\frac{2}{3}\right) \equiv (-1)^{\frac{1}{8}(3^2-1)} = -1 \tag{3.45}$$

since 3 is odd prime number. This means that equation (3.44) is unsolvable, i.e. 2 is non quadratic residue (mod 3). This is a contradiction.

Therefore, $3u$ can not be an element of $P_{+5}$ for $u \in Z$.

**(ii)** In a similar manner, suppose that $r$ is an element of set $P_{+5}$ and $7t, (t \in Z)$ is also an element of set $P_{+5}$ then

$$7tr + 5 = B^2 \tag{3.46}$$

has to satisfy for integer $B$. Applying (modulo 7) of both sides, we get

$$B^2 \equiv 5 \; (mod\; 7) \tag{3.47}$$

We have to calculate the Legendre symbol $\left(\frac{5}{7}\right)$ by using Theorem 2.1 and Definition 2.2. From Theorem 2.1, we obtain

$$\left(\frac{5}{7}\right)\left(\frac{7}{5}\right) = (-1)^{\frac{5-1}{2} \cdot \frac{7-1}{2}} = +1 \tag{3.48}$$

since 5 and 7 are odd primes. By substituting $\left(\frac{7}{5}\right) = \left(\frac{2}{5}\right) = -1$ into the (3. 48) then we have $\left(\frac{5}{7}\right) = -1$ which means that equation (3.47) is unsolvable. So $7t$ can not be an element of $P_{+5}$ for $t \in Z$.

**(iii)** Similarly, suppose that $r$ is an element of set $P_{+5}$ , If $13n, (n \in Z)$ is also an element of set $P_{+5}$ then

$$13nr + 5 = C^2 \tag{3.49}$$

has to satisfy for integer $C$. Applying (modulo 13) of both sides, we have

$$C^2 \equiv 5 \; (mod\; 13) \tag{3.50}$$

By using Theorem 2.1 and Definition 2.2, then we obtain

$$\left(\frac{5}{13}\right)\left(\frac{13}{5}\right) = (-1)^{\frac{5-1}{2} \cdot \frac{13-1}{2}} = +1 \tag{3.51}$$

since 5 and 13 are odd primes. By substituting $\left(\frac{13}{5}\right) = \left(\frac{3}{5}\right) = -1$ into the (3.51) then we have $\left(\frac{5}{13}\right) = -1$ which implies that equation (3.50) is unsolvable. So $13n$ can not be an element of $P_{+5}$ for $n \in Z$.

**Remark 3.9.** We can prove that there is no set $P_{+7}$ contains any positive multiple of 4,5 or 11 and there is no set $P_{-7}$ contains any positive multiple of 6,13 or 17 by using the similar way of the proof of the Theorem 3.7 or Theorem 3.8.

**REFERENCES**

**[1]** Anglin W.S., The queen of mathematics-An introduction to number theory, Kluwer Academic Publishers, Dordrecht, 1995.

**[2]** Baker A., Davenport H., 1969, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$ , Quarterly Journal of Mathematics, Oxford(2), 20, 129-137, 1969.

**[3]** Brown LE., Sets in Which $xy + k$ is Always a Square, Math. Comp, 45, 613-620, 1985.

**[4]** Dickson LE., History of Theory of Numbers and Diophantine Analysis, Vol 2, Dove Publications, New York, 2005.

**[5]** Filipin L A., Fujita Y., M. Mignotte, The non-extendibility of some parametric families of D(-1)-triples, Q. J. Math. 63, 605-621, 2012.

**[6]** Gopalan M. A., Vidhyalakshmi S., Mallika S., Some special non-extendable Diophantine triples, Sch. J. Eng. Tech. 2, 159-160, 2014.

**[7]** Grinstead C.M., On a Method of Solving a Class of Diophantine Equations . Math. Comp.,32, 936-940, 1978.

**[8]** Kanagasabapathy P., Ponnudurai T., The Simultaneous Diophantine Equations $y^2 - 3x^2 = -2$ and $z^2 - 8x^2 = -7$ , Quarterly Journal of Mathernatics, Oxford Ser (3), 26, 275-278, 1975.

**[9]** Katayama S., Several methods for solving simultaneous Fermat-Pell equations, J. Math. Tokushima Univ., 33 , 1–14, 1999.

**[10]** Kaygısız K., Şenay H., Contructions of Some New Nonextandable $P_k$ Sets, International Mathematical Forum, 2, no. 58, 2869 – 2874, 2007.

**[11]** Masser D.W., Rickert J.H., Simultaneous Pell Equations ,Number Theory , 61, 52-66, 1996.

**[12]** Mohanty P., Ramasamy A.M.S., The Simultaneous Diophantine Equations $5y^2 - 20 = x^2, 2y^2 + 1 = z^2$,.J.Number Theory.18,365-359, 1984.

**[13]** Mordell LJ., Diophantine Equations, Academic Press, New York, 1970.

**[14]** Mollin R.A., Fundamental Number theory wiyh Applications, CRC Press, 2008.

**[15]** Tzanakis N., Effective solution of two simultaneous Pell equations by the elliptic logarithm method, Acta Arithm., 103, 119–135, 2002.