CHARACTERIZATIONS OF SOME ASSOCIATED AND SPECIAL CURVES TO TYPE-2 BISHOP FRAME IN E^3

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Abstract:

In this paper, we investigate associated curves according to type-2 Bishop frame in E^3. In addition, necessary and sufficient conditions for a curve to be a regular one are studied to the mentioned frame. Finally we give characterization of the are length of spherical indicatrice and inclined curve using harmonic curvature.

Key Words: Spherical indicatrix, type-2 Bishop Frame, associated curve, harmonic curvature, inclined curve.

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Özet:

Biz, bu makalede, 3-boyutlu Öklid uzayında 2. Tip Bishop çatısına göre bağlantılı eğrileri araştırıyoruz. Bununla beraber, 2. Tip Bishop çatısına göre bir eğrinin regüler olmasının gerek ve yeter koşullarını inceliyoruz. Son olarak, harmonik eğrilikden yararlanarak küresel göstergelerle, inclined eğrilerinin yay uzunluklarına dair karakterizasyonlar veriyoruz.

1 INTRODUCTION

Curves as geometical objects are one of the fundamental structures of differential geometry. An increasing interest of the theory of curves makes researches of special curves a development. The curves are said to be associated curves which are obtained from the differential and geometrical relation between two or more curves. Associated curves are used in science and engineering. For instance , some of these curves are Bertrand curves, Mannheim curves, inclined curves, etc. There are many works on these curves (see [1], [2], [3], [8], [10]).

In this paper, we give some new characterizations of associated curves by using type-2 Bishop frame in E^3. Later, we characterize spherical indicatrices and inclined curves using harmonic curvatures.

2 PRELIMINARIES

The Euclidean 3-space $\mathbb{R}^3$ proved with the standard flat metric given by

$$<,>= dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}^3$. Recall that the norm of an arbitrary vector $\alpha\epsilon\mathbb{R}^3$ is given by $\|\alpha\| = \sqrt{<\alpha,\alpha>}$. α is called a unit speed curve if velocity vector υ of α is satisfied by $\|\upsilon\| = 1$.

Let X be a smooth vector field on M. We express that a smooth curve $w: I \to M$ is an integral curve of X if

$$w'(s) = X_{w(s)} \tag{1}$$

holds for any $s\epsilon I$.

Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve w in the space E^3. For an arbitrary curve w with the first and second curvatures κ and τ in the space E^3, the following Frenet-Serret formulae are given

$$\begin{cases} T'(s) = \kappa(s)N(s) \\ N'(s) = -\kappa(s)T(s) + \tau(s)B\,(s) \\ B'(s) = -\tau(s)N(s) \end{cases} \tag{2}$$

where the curvature functions are defined by $\kappa = \kappa\,(s) = \|T'(s)\|$ and

$\tau\,(s) = -< B', N >$, $B = -\int N(s)\tau(s)ds$. In the rest of paper , we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to define a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of orthonormal frame along a curve simply by parallel transporting each component of the frame.

Now ,we define some associated curves of a curve w in E^3 defined according to type-2 Bishop frame. For a Frenet curve $w: I \rightarrow E^3$, consider a vector field v given by

$$v(s) = \alpha(s)T(s) + \beta(s)N(s) + \gamma(s)B(s), \tag{3}$$

where α, β and γ are functions on I satisfying $\alpha^2(s) + \beta^2(s) + \gamma^2(s) = 1$. Then, an integral curve $\overline{w}(s)$ of v defined on I is a unit speed curve in E^3, [8].

The type-2 Bishop frame is expressed as (see, [3])

$$\begin{bmatrix} \xi_1' \\ \xi_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} . \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}$$

or

$$\begin{cases} \xi_1' = -\varepsilon_1 B, \\ \xi_2' = -\varepsilon_2 B, \\ B = \varepsilon_1 \xi_1 + \varepsilon_2 \xi_2. \end{cases} \tag{4}$$

In order to investigate a type-2 Bishop frame relation with Serret-Frenet frame, first we write

$$B = -\tau N = \varepsilon_1 \xi_1 + \varepsilon_2 \xi_2 \tag{5}$$

Taking the norm of both sides (5) , we have

$$\kappa(s) = \frac{d\theta(s)}{ds}, \tau(s) = \sqrt{\varepsilon_1^2 + \varepsilon_2^2} \tag{6}$$

Moreover, we may express

$\varepsilon_1(s) = -\tau\cos\theta(s)$, $\varepsilon_2(s) = -\tau\sin\theta\,(s)$. (7)

By this way , we conclude with $\theta(s) = arc\tan\frac{\varepsilon_2}{\varepsilon_1}$. The frame $\{\xi_1, \xi_2,$B$\}$ is properly oriented, and τ and $\theta(s) = \int_0^s K(s)ds$ are polar coordinates for the curve $w(s)$.

We write the tangent vector according to the frame $\{\xi_1, \xi_2, B\}$ as

$$T = \sin\theta(s)\xi_1 - \cos\theta(s)\xi_2$$

and differentiate with respect to s, we have

$$T' = \kappa N = \theta'(s)(\cos\theta(s)\xi_1 + \sin\theta\,(s)\xi_2) + (\sin\theta(s)\xi_1{}' - \cos\theta(s)\xi_2{}'). \quad (8)$$

Substituting $\xi_1' = -\varepsilon_1 B$ and $\xi_2' = -\varepsilon_2 B$ into (8) we have

$$\kappa N = \theta'(s)(\cos\theta(s)\xi_1 + \sin\theta(s)\xi_2)$$

In the above equation let us take $\theta'(s) = \kappa(s)$. Hence we immediately arrive at

$$N = \cos\theta(s)\xi_1 + \sin\theta(s)\xi_2.$$

Considering the obtained equations , the relation matrix between Serret-Frenet and the type-2 Bishop frame can be expressed [3]

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix} \quad (9)$$

Definition 2.1. The function

$$H = \frac{\varepsilon_2(s)}{\varepsilon_1(s)}$$

is called harmonic curvature function of the curve α provided that $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$, according to type-2 Bishop frame in Euclidean space.

Definition 2.2. The following differentiable fuction defined in an open interval

$I = \{t: a < t < b\}$ in the set of real numbers

$\alpha: I \to E^3$

$t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

is a curve in E^3 and if the first derivative of the curve is non-zero everywhere, so this curve is called a regular curve.

Definition 2.3. Let $\alpha = \alpha(s)$ be a regular curve with curvatures κ and τ . α is an inclined curve if and only if $\frac{\kappa}{\tau}$ =constant [5].

Remark 2.1. A principal-direction (resp. the binormal-direction) curve is an integral curve of $V(s)$ with $\alpha(s) = \gamma(s) = 0$, $\beta(s) = 1$ (resp. $\alpha(s) = \beta(s) = 0$, $\gamma(s) = 1$) for all s, see [8].

Proposition 2.1. [8], Let w be a curve in E^3 and let $\overline{w}$ be an integral curve of w. Then, the principal-direction curve of $\overline{w}$ equals to w up to the translation if and only if

$$\alpha(s) = 0, \beta(s) = -\cos(\int \tau(s)ds) \neq 0, \gamma(s) = \sin(\int \tau(s)ds). \tag{10}$$

For the rest of this paper, we assume that $\bar{s} = s$ without loss of generality.

3 ξ_1 −DIRECTION CURVE AND ξ_1 −DONOR CURVE; ξ_2 −DIRECTION CURVE AND ξ_2 −DONOR CURVE ACCORDING TO TYPE-2 BISHOP FRAME IN E^3

Definition 3.1. Let X be a curve in E^3. An integral curve of ξ_1 is called ξ_1 −direction curve of X according to type-2 Bishop frame if ξ_1 −direction curve is an integral curve of (11) with $\gamma(s) = \beta(s) = 0, \alpha(s) = 1$.

Definition 3.2. Let X be a curve in E^3. An integral curve of ξ_2 is called ξ_2 −direction curve of X according to type-2 Bishop frame if ξ_2 −direction curve is an integral curve of

$$v(s) = \alpha(s)\xi_1(s) + \beta(s)\xi_2(s) + \gamma(s)B(s) \tag{11}$$

with $\alpha(s) = \gamma(s) = 0 \, , \beta(s) = 1$.

Definition 3.3. An integral curve of ξ_1 is called ξ_1 −donor curve of X according to type-2 Bishop frame.

Definition 3.4. An integral curve of ξ_2 is called ξ_2 −donor curve of X according to type-2

Bishop frame.

Theorem 3.1. Let X be a curve in E^3 with the curvature κ and the torsion τ and $\bar{X}$ be the ξ_2 −Direction curve of X with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the Frenet frames of $\bar{X}$ are $\bar{T}(s) = \xi_2(s)$,$\bar{N}(s) = -B(s), \bar{B}(s) = \xi_1(s)$ and also the curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of $\bar{X}$ are given by $\bar{\kappa}\,(s) = \varepsilon_2(s)$ and $\bar{\tau}\,(s) = -\varepsilon_1(s)$.

Proof. Obviously, from the Definition 3.2, we write

$$\bar{X}' = \bar{T}(s) = \xi_2(s). \tag{12}$$

If we take the norm of the derivative of (12), then we obtain ,

$$\bar{\kappa}\,(s) = \varepsilon_2(s) \tag{13}$$

for $\varepsilon_2 > 0$.

Differentiating (12) with respect to s , we have

$$\bar{N}(s) = -B(s). \tag{14}$$

Taking the vector product of $\bar{T}$ and $\bar{N}$, we obtain

$$\bar{B}(s) = \xi_1(s), \tag{15}$$

and differentiating (15), we get

$$\bar{B}' = -\bar{\tau}\,\bar{N} = -\varepsilon_1 B, \tag{16}$$

substituting $\bar{N} = -B$ into (16) we find

$$\bar{\tau}\,(s) = -\varepsilon_1(s). \tag{17}$$

Corollary 3.1. Let X be a Frenet curve in E^3 with the curvature κ and the torsion τ and let $\bar{X}$ be the ξ_2 −direction curve of X with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the type-2 Bishop frame of $\bar{X}$ is given by

$$\begin{cases} \bar{T}(s) = \sin(\int \xi_2(s)ds)\bar{\xi}_1(s) - \cos(\int \xi_2(s)ds)\bar{\xi}_2(s), \\ \bar{N}(s) = \cos(\int \xi_2(s)ds)\bar{\xi}_1(s) + \sin(\int \xi_2(s)ds)\bar{\xi}_2(s), \\ \qquad \bar{B}(s) = \xi_1(s). \end{cases} \tag{18}$$

Proof. It is seen straightforwardly by using (9).

Corollary 3.2. If a curve X in E^3 is a ξ_2 −donor curve of a curve $\bar{X}$ with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$, then the torsion τ of the curve X is given by

$$\tau(s) = \sqrt{\bar{\kappa}^2(s) + \bar{\tau}^2(s)} \tag{19}$$

Proof. If we take the squares of (14) and (17), then we have (19).

Corollary 3.3. Let X be a curve in E^3 with the curvature κ and the torsion τ and let $\bar{X}$ be the ξ_2 −direction curve of X with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, it satisfies

$$\frac{\bar{\tau}(s)}{\bar{\kappa}(s)} = -\cot\theta(s).$$

Proof. It is seen straightforwardly.

Theorem 3.2. Let X be a curve in E^3 and let $\bar{X}$ an integral curve of (11). Then, the principal-direction curve of $\bar{X}$ equals to X up to the translation if and only if $\alpha = 0$ and

$$\alpha(s) = -\int \varepsilon_1(s)\gamma(s)ds.$$

Proof. Differentiating $\alpha^2(s) + \beta^2(s) + \gamma^2(s) = 1$ with respect to s, we get

$$\alpha\alpha' + \beta\beta' + \gamma\gamma' = 0 \tag{20}$$

Similarly differentiating (11) with respect to s, we have

$$v' = (\alpha' + \gamma\varepsilon_1)\xi_1 + (\beta' + \gamma\varepsilon_2)\xi_2 + (\gamma' - \alpha\varepsilon_1 - \beta\varepsilon_2)B,$$

since $v'(s) = \overline{X''}(s) = \overline{T}'(s) = \bar{\kappa}\bar{N}(s)$, X is a principal-direction curve of $\bar{X}$, ie.,

$\bar{X}(s) = T(s) = \bar{N}$ if and only if

$$\begin{cases} \alpha' + \gamma\varepsilon_1 \neq 0, \\ \beta' + \gamma\varepsilon_2 = 0, \\ \gamma' - \alpha\varepsilon_1 - \beta\varepsilon_2 = 0 \end{cases} \tag{21}$$

hold. Multiplying the third equation $(21)_3$ with γ and the second equation in (20) with β, we have $\alpha(\alpha' + \varepsilon_1\gamma) = 0$. Since $\alpha' + \varepsilon_1\gamma \neq 0$, if it follows that we get $\alpha = 0$ and

$$\alpha(s) = -\int \varepsilon_1(s)\gamma(s)ds.$$

Theorem 3.3. Let X be a curve with the curvature κ and the torsion τ in E^3 and let $\bar{X}$ be the ξ_1 −direction curve of X be the ξ_1 −direction curve of X with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the Frenet frames of $\bar{X}$ are $\bar{T}(s) = \xi_1(s), \bar{N}(s) = -B(s), \bar{B}(s) = \xi_2(s)$ and curvature $\bar{\kappa}$ and torsion $\bar{\tau}$ of X are determined by

$\bar{\kappa}(s) = \varepsilon_1(s)$ and $\bar{\tau}(s) = -\varepsilon_2(s)$.

Proof. From the Definition 3.1, we can easily obtain that

$$\bar{X}'(s) = \bar{T}(s) = \xi_1(s). \tag{22}$$

If we take the norm of the derivative of (22), then we obtain,

$$\bar{\kappa}(s) = \varepsilon_1(s) \tag{23}$$

for $\varepsilon_1 > 0$. Differentiating of (22) , we get

$$\bar{N}(s) = -B(s). \tag{24}$$

Taking the vector product of $\bar{T}$ and $\bar{N}$, we have

$$\bar{B} = \xi_2(s). \tag{25}$$

Differentiating (25) we obtain

$$\bar{B}' = -\varepsilon_2 B, \tag{26}$$

and substituting (24) into (26) we find

$$\bar{\tau}(s) = -\varepsilon_2(s). \tag{27}$$

Corollary 3.4. Let X be a curve in E^3 with the curvature κ and the torsion τ and let $\bar{X}$ be the ξ_1 −direction curve of X with the curvature $\bar{\kappa}$ and the torsion $\bar{\tau}$. Then, the type-2 Bishop frame of $\bar{X}$ is given by

$$\begin{cases} \bar{T}(s) = \sin(\int \xi_1(s)ds)\bar{\xi_1}(s) - \cos(\int \xi_1(s)ds)\bar{\xi_2}(s), \\ \bar{N}(s) = \cos(\int \xi_1(s)ds)\bar{\xi_1}(s) + \sin(\int \xi_1(s)ds)\bar{\xi_2}(s), \\ \qquad \bar{B}(s) = \xi_2(s). \end{cases}$$

Proof. It is seen straightforwardly by using (9).

Corollary 3.5. If a curve X in E^3 is a ξ_1 −donor curve of a curve X with the curvature $\bar{\kappa}$ and the torsion $\overline{\tau}$, then the torsion τ of the curve X is given by

$$\tau(s) = \sqrt{\bar{\kappa}^2(s) + \overline{\tau}^2(s)} \qquad (28)$$

Proof. If we take the squares of (23) and (27), then we have (28).

Corollary 3.6. Let X be a curve in E^3 with the curvature κ and the torsion τ and let $\bar{X}$ be the ξ_1 −direction curve of X with the curvature $\bar{\kappa}$ and the torsion $\overline{\tau}$. Then, it satisfies

$$\frac{\overline{\tau}(s)}{\bar{\kappa}(s)} = -\tan\theta(s).$$

Proof. It is seen straightforwardly.

4 CHARACTERIZATION OF SLANT HELICES ACCORDING TO TYPE-2 BISHOP FRAME IN E^3

Let us denote the tangent, the principal normal, and the binormal indicatrices of curve α with α_1, α_2 and α_3 respectively. The following properties must be taken into the consideration for spherical indicatrix of curve α to be a regular curve

i) The curve α_1 is regular $\Leftrightarrow \left\|\frac{d\alpha_1}{ds}\right\| = \varepsilon_1 \neq 0$, since $\alpha_1 = \xi_1 \Rightarrow \frac{d\alpha_1}{ds} = -\varepsilon_1 B$

ii) Similarly the curve α_2 is regular $\Leftrightarrow \left\|\frac{d\alpha_2}{ds}\right\| = \varepsilon_2 \neq 0$, since

$\alpha_2 = \xi_2 \Rightarrow \frac{d\alpha_2}{ds} = -\varepsilon_2 B.$

iii) Also , curve α_3 is regular $\Leftrightarrow \left\|\frac{d\alpha_1}{ds}\right\| = \sqrt{\varepsilon_1^2 + {\varepsilon_2}^2}$ $(\varepsilon_1 \neq 0\,, \varepsilon_2 \neq 0)$ since

$\alpha_3 = B \Rightarrow \frac{d\alpha_3}{ds} = \varepsilon_1\xi_1 + \varepsilon_2\xi_2.$

4.1 The arc-length of the tangent indicatrices of the curve α

Let $\xi_1(s) = \xi(s)$ be the tangent vector field of the curve

$\alpha: I \subset E \to E^3$

$s \to \alpha(s)$

The spherical curve of α_{ξ_1}=$\vec{\xi}$ on S^2 is called tangent indicatrices of α. Let s be the arc-length parameter of α. If we denote the arc-length of the curve α_{ξ_1} by s_{ξ_1} , then we can write

$\alpha_{\xi_1}\left(s_{\xi_1}\right) = \overrightarrow{\xi_1}(s)$. Letting $\frac{d\alpha_{\xi_1}}{ds_{\xi_1}} = \xi_{1_{\xi_1}}$ we have $T_{\xi_1} = (-\varepsilon_1 B)\frac{ds}{ds_{\xi_1}}$. Hence, we obtain $\frac{dS_{\xi_1}}{ds} = \varepsilon_1$. Since the harmonic curvature of α is $H = \frac{\varepsilon_2}{\varepsilon_1}$, then we get $S_{\xi_1} = \int \frac{\varepsilon_2}{H} ds + c$.

4.2 The arc-length of the principal normal indicatrices of the curve α

Let $\xi_2 = \overrightarrow{\xi_2}(s)$ be a principal normal vector field of the curve

$\alpha\colon \mathrm{I} \subset E \to E^3$

$s \to \alpha(s)$

The spherical curve $\alpha_{\xi_2} = \overrightarrow{\xi_2}$ on S^2 is called principal spherical indicatrices for α. Let $s \in \mathrm{I}$ be the arc-length of α. If we denote the arc-length of α_{ξ_2}, by S_{ξ_2}we may write

$\alpha_{\xi_2}\left(S_{\xi_2}\right) = \overrightarrow{\xi_2}(s)$. Moreover, letting $\frac{d\alpha_{\xi_2}}{dS_{\xi_2}} = \xi_{1_{\xi_2}}$, we have $T_{\xi_2} = (-\varepsilon_2 B)\frac{ds}{ds_{\xi_2}}$. Hence we get

$\frac{dS_{\xi_2}}{ds} = \varepsilon_2$.

If the harmonic curvature of α is $H = \frac{\varepsilon_2}{\varepsilon_1}$, then we get $S_{\xi_2} = \int \frac{\varepsilon_2}{H} ds + c$.

4.3 The arc-length of the binormal indicatrices of the curve α

Let $\vec{B} = \vec{B}(s)$ be the binormal vector field of the curve

$\alpha\colon \mathrm{I} \subset E \to E^3$

$s \to \alpha(s)$

The spherical curve $\alpha_B = \vec{B}$ on S^2 is called binormal indicatries of α . Let $s \in \mathrm{I}$ be the arc-length parameter of α. If we denote the arc-length parameter of α_B by s_B, we may write $\alpha_B(s_B) = \vec{B}(s)$.

Moreover, letting $\frac{d\alpha_B}{ds_B} = T_B$, we obtain $T_B = (\varepsilon_1\xi_1 + \varepsilon_2\xi_2)\frac{ds}{ds_B}$. Obviously, we end up

$\frac{ds_B}{ds} = \sqrt{\varepsilon_1^2 + \varepsilon_2^2} = \tau.$

In this case, we give the following result. If τ is the second curvature of the curve $\alpha : \mathrm{I} \to E^3$, then the arc-length s_B of the binormal α_B of α is $s_B = \int \tau ds$.

If the harmonic curvature of α is $H = \frac{\varepsilon_2}{\varepsilon_1}$, then we get $s_B = \int \varepsilon_1 \sqrt{1 + H^2}\, ds$.

Thus we can give the following theorem:

Theorem 4.1. If the curve $\alpha \subset E^3$ is an inclined curve (general helix), then H is constant.

Proof: From the equation (7), then we can write

$$\frac{\varepsilon_2}{\varepsilon_1} = \tan\theta. \tag{29}$$

Differentiating with respect to s we have

$$(\frac{\varepsilon_2}{\varepsilon_1})' = (1 + \tan^2\theta)\frac{d\theta}{ds} \text{ or}$$

$$(\frac{\varepsilon_2}{\varepsilon_1})' = [1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2]\frac{d\theta}{ds}.$$

Rearrangement of this equation, we get

$$\frac{d\theta}{ds} = \frac{(\frac{\varepsilon_2}{\varepsilon_1})'}{1+(\frac{\varepsilon_2}{\varepsilon_1})^2}, \tag{30}$$

and using $H = \frac{\varepsilon_2}{\varepsilon_1}$ in (30), we obtain

$$\frac{d\theta}{ds} = \frac{H'}{1+H^2} \tag{31}$$

integrating (31), we find

$$\theta = \int \frac{H'}{1+H^2} ds,$$

since $H' = \frac{dH}{ds}$ implies $H'ds = dH$, then we have $\theta = arctan\, H + c$, where c is a constant.

If the curve α is an inclined curve, then from (29) θ is constant, i.e.,

$$arctanH = \theta - c = constant. \tag{32}$$

Rearranging (32), we have $H = \tan(\theta - c) = constant$.Hence, the proof is completed as required.

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