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Contributions to differential geometry of isotropic curves in the complex space $\mathbb{C}^3$ – II

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ABSTRACT

This study investigates classical differential geometry of isotropic curves in the complex space $\mathbb{C}^3$. First, we deal with spherical images of isotropic curves, and then obtain some results regarding these curves. Therefore, we continue to study these spherical indicatrices as Darboux curves and Bertrand mates. Also, we examine isotropic slant helices in $\mathbb{C}^3$. Additionally, we show that the vectors of isotropic curves and their pseudo-curvatures satisfy a vectorial differential equation of the second order with variable coefficients. We study this differential equation under some special cases. Finally, we give the conditions for an isotropic curve to be Darboux helix in $\mathbb{C}^3$. Next, we define the constant breadth of isotropic curves and express some characterizations of these curves in terms of E. Cartan equations in $\mathbb{C}^3$.

1. Introduction

It is observed that the imaginary curves in the complex space were pioneered by E. Cartan. Cartan defined the moving frame of an imaginary curve and its special equations in $\mathbb{C}^3$. In [3], the Cartan equations of isotropic curve were extended to the four dimensional complex space $\mathbb{C}^4$. Moreover, Pekmen gave some characterizations of minimal curves by means of E. Cartan equations in $\mathbb{C}^3$ [13]. Also Şemin had mentioned the complex elements and complex curves in the real space $\mathbb{R}^3$ [16].

In the complex space $\mathbb{C}^3$, helices were characterized by [19]. In complex space $\mathbb{C}^4$, Yılmaz characterized the isotropic curves with constant pseudo-curvature which is called the slant isotropic helix [17]. Yılmaz and Turgut gave some properties of isotropic helices in $\mathbb{C}^3$ [19]. Recently, the representation formula for an isotropic curve with pseudo arc length parameter and the structure function of such curves were defined by Qian and they characterized the isotropic Bertrand curve and k-type isotropic helices by using the representation and the Frenet formulas [15].

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Several authors introduced different types of helices and investigated their properties. For instance, Barros et al. studied general helices in 3-dimensional Lorentzian space [4]. Izumiya and Takeuchi defined slant helices by the property that principal normal makes a constant angle with a fixed direction [7]. Recently, Körpınar, and Körpınar et al. studied characterization of minimizing energy of biharmonic particles in spacetime [8,9]. Kula and Yaylı studied spherical images of tangent and binormal indicatrices of slant helices and also showed that spherical images are spherical helices [10]. Ali and Lopez gave some characterizations of slant helices in Minkowski 3-space $\mathbb{E}_1^3$ [2]. Pekmen and Paşalı characterized unit speed spacelike curves whose images lie on a Lorentzian sphere in Minkowski 3-space $\mathbb{E}_1^3$ [14].

The Darboux rotation axis was introduced for a space curve by Barthel [5]. Afterwards, the results obtained for Euclidean space in the work of Barthel were studied by Yücesan for a Lorentzian space curve [20]. The Darboux vector of isotropic curves was introduced by Semin. Curves of constant breadth were introduced by Euler [6]. Variable space is studied by these special curves [12,11,18].

In this work, using not common vector field as Cartan frame, we introduce a new spherical image and a Darboux helix in $\mathbb{C}^3$. Also translating Cartan frame's vector fields to the center of sphere, we obtain the spherical indicatrices of isotropic curves. Moreover, we investigate the Darboux vector and Darboux helices in $\mathbb{C}^3$. Additionally, we study the constant breadth of isotropic curves in the same space.

2. Preliminaries

The three dimensional complex space $\mathbb{C}^3$ is given with the standard flat metric as follows:

$$\langle , \rangle = dx_2^2 + 2dx_1dx_3,$$

where (x_1, x_2, x_3) is a complex coordinate system of $\mathbb{C}^3$.

Let x_p be a complex analytic function of a complex variable t. Then the vector function

$$\vec{x}(t) = \sum_{p=1}^{3} x_p(t)\vec{k}_p \tag{1}$$

is called an imaginary curve, where $t = t_1 + it_2$, $\vec{x} : \mathbb{C} \longrightarrow \mathbb{C}^3$ and $\vec{k}_p$ are standard basis unit vectors of $\mathbb{E}^3$ [1,16].

In this space, a vector which has a minimal direction is called an isotropic vector or minimal vector, that is a vector u is a minimal vector if and only if $u^2 = 0$ [16]. The curves, of which the square of the distance between the two points is equal to zero, are called minimal or isotropic curves [19]. Let s denote pseudo-arclength, a curve is a minimal (isotropic) curve if and only if $ds^2 = 0$, where s denotes the pseudo arc-length. Thus it is obvious that an isotropic curve satisfies vectorial differential equation

$$[\vec{x}'(t)]^2 = 0, \tag{2}$$

where $\dfrac{dx}{dt} = \vec{x}'(t) \neq 0$.

For each point $\vec{x}$ of the isotropic curve, E. Cartan frame is defined (for well-known complex number $i^2 = -1$) as follows, see [1,16].

$$\begin{aligned} \vec{e}_1 &= \vec{x}', \\ \vec{e}_2 &= i\vec{x}'', \\ \vec{e}_3 &= -\frac{\beta}{2}\vec{x}' + \vec{x}''', \end{aligned} \tag{3}$$

where $\beta = (\vec{x}''')^2$. The moving E. Cartan frame along the isotropic curve $\vec{x}$ in $\mathbb{C}^3$ is given by (3) which is denoted by $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. The inner products of these frame vectors are given by

$$\vec{e}_j.\vec{e}_k = \begin{cases} 0 & \text{if } j+k = 1,2,3;\ \text{Mod}(4), \\ 1 & \text{if } j+k = 4. \end{cases} \tag{4}$$

The vector and mixed products of these frame vectors are given by

$$\vec{e}_j \times \vec{e}_k = i\,\vec{e}_{j+k-2}, \quad \vec{e}_1.(\vec{e}_2 \times \vec{e}_3) = i,$$

for $j, k = 1, 2, 3$. The pseudo-arclength

$$s = \int_{t_0}^{t} -[(\vec{x}'')^2]^{\frac{1}{4}} dt$$

is an invariant with respect to parameter t [16]. Thus the vectors $\vec{e}_1$ and $\vec{e}_3$ are isotropic vectors, while $\vec{e}_2$ is a real vector. E. Cartan derivative formulas can be deduced from (3) as follows

$$\begin{aligned} \vec{e}_1' &= -i\,\vec{e}_2, \\ \vec{e}_2' &= i(\kappa\vec{e}_1 + \vec{e}_3), \\ \vec{e}_3' &= -i\kappa\vec{e}_2, \end{aligned} \tag{5}$$

where $\kappa = \dfrac{\beta}{2}$ is called pseudo-curvature of isotropic curve $\vec{x} = \vec{x}(s)$ [13]. These equations can be used if the minimal curve is at least of class C^4. Here $(')$ denotes derivative according to pseudo-arclength s. In the rest of the paper, we suppose that pseudo-curvature κ is non-vanishing except in the case of an isotropic cubic. Isotropic sphere with center $\vec{m}$ and radius $r > 0$ in $\mathbb{C}^3$ is defined [17] by

$$S^2 = \{\vec{p} = (p_1, p_2, p_3) \in \mathbb{C}^3 : (\vec{p} - \vec{m})^2 = 0\}.$$

Definition 2.1. An isotropic curve $\vec{x} = \vec{x}(s)$ in $\mathbb{C}^3$ is called an isotropic cubic if pseudo-curvature κ of $\vec{x}(s)$ is congruent to zero [17].

Definition 2.2. An isotropic curve $\vec{x} = \vec{x}(s)$ in $\mathbb{C}^3$ is called an isotropic helix if the tangent vector e_1 of $\vec{x}(s)$ is isotropic vector [16].

Let $\vec{x} = \vec{x}(s)$ be an isotropic curve with the pseudo-curvature $\kappa \neq 0$, the pseudo-Darboux vector of the curve is defined as

$$\vec{e}_q' = \vec{w}_0 \times \vec{e}_q \quad (q = 1, \ldots, 3).$$

If we write the pseudo-Darboux vector of the curve as follows,

$$\vec{w}_0 = \sum_{q=1}^{3} \vec{x}_q \vec{e}_q,$$

then we obtain

$$\vec{w}_0 = \kappa\vec{e}_1 - \vec{e}_3.$$

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The norm of Darboux vector of the curve $\vec{x} = \vec{x}(s)$ in $\mathbb{C}^3$ is defined as

$$\|\vec{\vec{w}}_0\| = \sqrt{(\kappa\vec{e}_1 - \vec{e}_3)^2} = i\sqrt{2\kappa}$$

which is called pseudo-Lancret curvature [16].

Definition 2.3. Let $\vec{x} = \vec{x}(s)$ be an isotropic curve in $\mathbb{C}^3$. If there exists another isotropic curve $\vec{x}^*(s)$ in $\mathbb{C}^3$ such that principal normal vector field of $\vec{x}^*(s)$ coincides with that of $\vec{x} = \vec{x}(s)$, then the curve $\vec{x}(s)$ is called an isotropic Bertrand curve, and $\vec{x}^*(s)$ is called the isotropic Bertrand mate of $\vec{x}(s)$ and vice versa [15].

3. Some characterizations of spherical indicatrices of isotropic curves in $\mathbb{C}^3$

In this section, first we give some new characterizations of spherical indicatrices of isotropic curves in $\mathbb{C}^3$. Then, we continue to study these spherical indicatrices as Darboux curves and Bertrand mates. Also, we examine isotropic slant helices in $\mathbb{C}^3$.

3.1. Spherical indicatrices of isotropic curves in $\mathbb{C}^3$

Definition 3.1. Let $\alpha = \alpha(s)$ be a regular isotropic curve in $\mathbb{C}^3$. If we translate the first vector field $\vec{e}_1$ of E. Cartan frame to the center O of the unit isotropic sphere S^2, then we obtain spherical image $\varphi = \varphi(s_{\varphi\xi})$. This curve is called $\widetilde{E}_1$ spherical image or indicatrix of the isotropic curve $\alpha = \alpha(s)$.

Theorem 3.1. *Let α be a unit isotropic curve and $\widetilde{E}_1$ be a complex unit speed curves in $\mathbb{C}^3$ and $\widetilde{E}_1$ be a first spherical image of the isotropic curve α. The E. Cartan apparatus of $\widetilde{E}_1$ ($\{e_{1\varphi}, e_{2\varphi}, e_{3\varphi}, \kappa_\varphi\}$) can be formed according to E. Cartan apparatus of α ($\{e_1, e_2, e_3, \kappa\}$).*

Proof. Let $\varphi = \varphi(s_\varphi)$ be the spherical image $\widetilde{E}_1$ of a regular isotropic curve $\alpha = \alpha(s)$. We shall investigate relations among the Cartan invariants of α and $\widetilde{E}_1$. First differentiating φ gives us

$$\varphi' = \frac{d\varphi}{ds_\varphi}.\frac{ds_\varphi}{ds} = -ie_2. \tag{6}$$

Here we shall denote differentiation according to s by a dash. Taking the norm of (6), we have

$$e_{1\varphi} = -e_2, \quad \frac{ds_\varphi}{ds} = i. \tag{7}$$

Differentiating (7) gives us

$$e'_{1\varphi} = \frac{de_{1\varphi}}{ds_\varphi}.\frac{ds_\varphi}{ds} = -i(\kappa e_1 + e_3),$$

so we have

$$\dot{e}_{1\varphi} = -(\kappa e_1 + e_3).$$

Hence we obtain the pseudo-curvature and principal normal of φ as

$$e_{2\varphi} = -i(\kappa e_1 + e_3), \quad \kappa_\varphi = -2\kappa^2. \tag{8}$$

(5)$_3$ gives us the binormal vector field of the spherical indicatrix $\widetilde{E}_1$ of the isotropic curve $\alpha = \alpha(s)$ as follows

$$e_{3\varphi} = -\kappa e_1 + 2\kappa i(\kappa + 1)e_2.$$

Corollary 3.1. *The spherical image $\widetilde{E}_1$ of a regular isotropic curve $\alpha = \alpha(s)$ is not an isotropic curve.*

Proof. The result is straightforwardly seen by $e_{1\varphi}$ which is not isotropic vector from (7).

Corollary 3.2. *Let $\varphi = \varphi(s_\varphi)$ be the spherical image $\widetilde{E}_1$ of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is constant, then the spherical indicatrix $\widetilde{E}_1$ of $\varphi = \varphi(s_\varphi)$ is a pseudo-helix in $\mathbb{C}^3$.*

Proof. Let $\varphi = \varphi(s_\varphi)$ be the spherical image $\widetilde{E}_1$ of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is constant in terms of equations (8)$_2$, then we have $\kappa_\varphi =$ constant. Therefore φ is a pseudo helix.

Theorem 3.2. *Let $\vec{x} = \vec{x}(s)$ be a pseudo-arc lengthed isotropic curve in $\mathbb{C}^3$. The second component of the position vector of the curve with respect to pseudo arc lengthed tangent spherical indicatrix satisfies a second order differential equation as*

$$\frac{d^2 e_2}{ds_\varphi^2} + 2i\kappa e_2 + \frac{1}{i}\frac{d\kappa}{ds_\varphi}\left(\int e_2 ds_\varphi\right) = 0. \tag{9}$$

Proof. We know that $\dfrac{ds_\varphi}{ds} = i$ from (7)$_2$. Differentiating Cartan derivative equation (5)$_1$ with respect to pseudo arc-lengthed parameter of tangent spherical image, we obtain

$$\frac{de_1}{ds_\varphi} = \frac{de_1}{ds}\frac{ds_\varphi}{ds} = (-ie_2)\frac{1}{i}. \tag{10}$$

Rearranging (10), we have

$$\frac{de_1}{ds_\varphi} = -e_2. \tag{11}$$

Similarly, if we take a derivative of (5)$_2$ and (5)$_3$, we obtain

$$\begin{aligned}\frac{de_2}{ds_\varphi} &= \frac{de_2}{ds}\frac{ds}{ds_\varphi} = \kappa e_1 + e_3,\\ \frac{de_3}{ds_\varphi} &= \frac{de_3}{ds}\frac{ds}{ds_\varphi} = -\kappa e_2.\end{aligned} \tag{12}$$

Thus differentiating Cartan derivative formulas with respect to pseudo-arc lengthed parameter of tangent spherical image, and using (11) and (12), we get

$$\begin{aligned}e_1' &= -e_2,\\ e_2' &= \kappa e_1 + e_3,\\ e_3' &= -\kappa e_2.\end{aligned} \tag{13}$$

From (13), we have

$$e_3 = e_2' - \kappa e_1. \tag{14}$$

And from derivative of (14) and $(13)_1$, we obtain the equation (9). The differential equation in (9) is also a characterization of the isotropic curve $\vec{x} = \vec{x}(s)$. The position vector of an arbitrary isotropic curve with respect to E. Cartan frame can be determined by means of its solution, however a general solution of (9) has not yet been found. Due to this, $\vec{x}(s)$ is an isotropic helix for explicit result. Let $\vec{x} = \vec{x}(s)$ be an isotropic cubic, then $\kappa = 0$. Therefore the differential equation in (9) turns into

$$\frac{d^2 e_2}{ds_\varphi^2} = 0. \tag{15}$$

As solution of equation (15), we have

$$e_2(s_\varphi) = c_0 + c_1 s_\varphi;\ c_0, c_1 \text{ being constants.}$$

Let $\vec{x} = \vec{x}(s)$ be an isotropic helix, then $\kappa =$ constant. Therefore, we have the differential equation in (9) as follows

$$\frac{d^2 e_2}{ds_\varphi^2} + 2i\kappa e_2 = 0. \tag{16}$$

As a solution of equation (16), we get

$$e_2(s_\varphi) = \alpha_1 e^{-\sqrt{2i\kappa} s_\varphi} + \alpha_2 e^{\sqrt{2i\kappa} s_\varphi}.$$

Definition 3.2. Let $\alpha = \alpha(s)$ be a regular isotropic curve in $\mathbb{C}^3$. If we translate the second vector field $\vec{e}_2$ of E. Cartan frame to the center O of the unit isotropic sphere S^2, we obtain a spherical image $\gamma = \gamma(s_\gamma)$. This complex curve is called $\widetilde{E}_2$ spherical image or indicatrix of the isotropic curve $\alpha = \alpha(s)$.

Theorem 3.3. *Let γ be a unit speed isotropic curve in $\mathbb{C}^3$ and $\widetilde{E}_2$ be a second spherical image of the isotropic curve α. The E. Cartan apparatus of $\widetilde{E}_2$ $(\{e_{1\gamma}, e_{2\gamma}, e_{3\gamma}, \kappa_\gamma\})$ can be formed according to E. Cartan apparatus of α $(\{e_1, e_2, e_3, \kappa\})$.*

Proof. Let $\gamma = \gamma(s_\gamma)$ be the $\widetilde{E}_2$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. We shall investigate relations among the Cartan invariants of α and $\widetilde{E}_2$. First differentiating γ gives us

$$\gamma' = \frac{d\gamma}{ds_\gamma}.\frac{ds_\gamma}{ds} = i(\kappa e_1 + e_3). \tag{17}$$

Similar to spherical image $\widetilde{E}_2$, one can have

$$e_{1\gamma} = \sqrt{\frac{\kappa}{2}} e_1 + \frac{1}{\sqrt{2\kappa}} e_3, \quad \frac{ds_\gamma}{ds} = i\sqrt{2\kappa}, \tag{18}$$

so differentiating $e_{1\gamma}$ in (18), we obtain

$$e_{1\gamma}' = \frac{de_{1\gamma}}{ds_\gamma}.\frac{ds_\gamma}{ds} = (\sqrt{\frac{\kappa}{2}})' e_1 - i\sqrt{2\kappa} e_2 + \frac{1}{\sqrt{2\kappa}} e_3$$

or in other words,

$$\dot{e}_{1\gamma} = \frac{1}{i\sqrt{2\kappa}}(\sqrt{\frac{\kappa}{2}})'e_1 - e_2 - \frac{i}{2\kappa}e_3.$$

Hence we express that

$$\kappa_\gamma = \{\frac{1}{\sqrt{\kappa}} + \frac{1}{2\kappa}(\sqrt{\frac{\kappa}{2}})'\} + \{-\sqrt{\kappa} - \frac{1}{\sqrt{2k}} + \frac{k}{2\sqrt{2}}\}\frac{i}{2},$$
$$e_{2\gamma} = \frac{i}{\sqrt{2\kappa}}(\sqrt{\frac{\kappa}{2}})'e_1 - ie_2 - e_3. \tag{19}$$

By the Cartan formula of the binormal vector, we have

$$e_{3\gamma} = [\frac{1}{2\kappa}(\sqrt{\frac{\kappa}{2}})' - \frac{\sqrt{\kappa}}{2} - \frac{\kappa_\gamma\kappa}{2}i]e_1 + [\frac{i}{\sqrt{2\kappa}} - \frac{\sqrt{\kappa}}{2}]e_2 + [-\frac{\kappa_\gamma}{2}i - \frac{1}{2\sqrt{\kappa}}]e_3. \tag{20}$$

Theorem 3.4. *Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. The second component of the Cartan frame of the curve with respect to pseudo arc lengthed normal spherical indicatrix satisfies a second order differential equation as follows:*

$$\frac{d^2e_2}{ds_\gamma^2} + \frac{1}{\sqrt{2\kappa}}\frac{d}{ds_\gamma}(\sqrt{2\kappa})\frac{de_2}{ds_\gamma} + i\sqrt{2\kappa}e_2 + (\int\frac{e_2}{\sqrt{2\kappa}}ds_\gamma)\frac{1}{\sqrt{2\kappa}}\frac{d\kappa}{ds_\gamma} = 0. \tag{21}$$

Proof. Let $\vec{x} = \vec{x}(s)$ be pseudo (isotropic) curve in $\mathbb{C}^3$. Since (18), we may write the Cartan frame of this curve with respect to pseudo arc-lengthed normal spherical indicatrix

$$\frac{de_1}{ds_\gamma} = \frac{-e_2}{\sqrt{2\kappa}}, \quad \frac{de_2}{ds_\gamma} = \frac{\kappa e_1 + e_3}{\sqrt{2\kappa}}, \quad \frac{de_3}{ds_\gamma} = \frac{-\kappa e_2}{\sqrt{2\kappa}}. \tag{22}$$

From $(22)_2$, we can easily obtain

$$e_3 = \sqrt{2\kappa}\frac{de_2}{ds_\gamma} - \kappa e_1. \tag{23}$$

Differentiating (23) and using $(22)_1$, we obtain the equation in (21).

We characterized the isotropic curve $\vec{x} = \vec{x}(s)$ using the differential equation (21). If $\kappa = 0$, then the equation (21) becomes undefined. In this case, the isotropic curve $\vec{x} = \vec{x}(s)$ can not be an isotropic cubic. If $\kappa =$ constant, then the equation (21) turns into the following form

$$\frac{d^2e_2}{ds_\gamma^2} + i\sqrt{2\kappa}e_2 = 0. \tag{24}$$

The solution of the equation (24) are obtained as follows:

$$e_2(s_\gamma) = \delta_1 e^{-i\sqrt[4]{2i\kappa}s_\gamma} + \delta_2 e^{i\sqrt[4]{2i\kappa}s_\gamma}.$$

Corollary 3.3. *The spherical image $\widetilde{E}_2$ of a regular isotropic curve $\alpha = \alpha(s)$ is not an isotropic curve.*

Proof. The result is straightforwardly seen by $e_{1\gamma}$ which is not isotropic vector from $(18)_1$.

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Corollary 3.4. *Let $\gamma = \gamma(s_\gamma)$ be the $\widetilde{E}_2$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is zero, then the $\widetilde{E}_2$ spherical indicatrix $\gamma = \gamma(s_\gamma)$ is not an isotropic cubic in $\mathbb{C}^3$.*

Proof. Let $\gamma = \gamma(s_\gamma)$ be the $\widetilde{E}_2$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the ratio of pseudo-curvatures of $\alpha = \alpha(s)$ is zero in terms of κ_γ in $(19)_1$, then we have κ_γ which is undefined. Therefore, γ is not an isotropic cubic.

Corollary 3.5. *Let $\gamma = \gamma(s_\gamma)$ be the $\widetilde{E}_2$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is constant, then the $\widetilde{E}_2$ spherical indicatrix $\gamma = \gamma(s_\gamma)$ is a pseudo-helix in $\mathbb{C}^3$.*

Proof. Let $\gamma = \gamma(s_\gamma)$ be the $\widetilde{E}_2$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is constant in terms of κ_γ in $(19)_1$, then we have $\kappa_\gamma =$ constant. Therefore, γ is a pseudo helix.

Definition 3.3. Let $\alpha = \alpha(s)$ be a regular isotropic curve in $\mathbb{C}^3$. If we translate the third vector field $\vec{e}_3$ of E. Cartan frame to the center O of the unit isotropic sphere S^2, then we obtain a spherical image $\xi = \xi(s_\xi)$. This complex curve is called $\widetilde{E}_3$ spherical image or indicatrix of the isotropic curve $\alpha = \alpha(s)$.

Theorem 3.5. *Let ξ be the unit speed isotropic curve in $\mathbb{C}^3$ and $\widetilde{E}_3$ be a third spherical image of the isotropic curve α. The E. Cartan apparatus of $\widetilde{E}_3$ $(\{e_{1\xi}, e_{2\xi}, e_{3\xi}, \kappa_\xi\})$ can be formed according to E. Cartan apparatus of α $(\{e_1, e_2, e_3, \kappa\})$.*

Proof. Let $\xi = \xi(s_\xi)$ be the $\widetilde{E}_3$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. First, differentiating ξ with respect to s gives us

$$\xi' = \frac{d\xi}{ds_\xi}.\frac{ds_\xi}{ds} = -i\kappa e_2.$$

In terms of Cartan frame vector fields, we immediately arrive at

$$e_{1\xi} = -e_2, \quad \frac{ds_\xi}{ds} = -i\kappa. \tag{25}$$

In order to determine pseudo-curvature of ξ, we write

$$-\dot{e}_{1\xi} = -e_1 + \frac{1}{i\kappa}e_3.$$

Hence, we immediately reach the following result

$$e_{2\xi} = -ie_1 + \frac{1}{\kappa}e_3, \quad \kappa_\xi = -\frac{2}{\kappa^2}. \tag{26}$$

By the Cartan formula of the isotropic binormal vector, we have

$$e_{3\xi} = -\frac{2}{\kappa}ie_1 + \frac{2}{\kappa}ie_2 + [-\frac{2}{\kappa^2}i + \frac{1}{i\kappa}(\frac{1}{\kappa})']e_3. \tag{27}$$

Corollary 3.6. *The spherical image $\widetilde{E}_3$ of a regular isotropic curve $\alpha = \alpha(s)$ is not an isotropic curve.*

Proof. The result is straightforwardly seen by $e_{1\xi}$ which is not isotropic vector from $(25)_1$.

Corollary 3.7. *Let $\xi = \xi(s_\xi)$ be the $\widetilde{E}_3$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is zero, then the $\widetilde{E}_3$ spherical indicatrix $\xi = \xi(s_\xi)$ is not an isotropic cubic in $\mathbb{C}^3$.*

Proof. Let $\xi = \xi(s_\xi)$ be the $\widetilde{E}_3$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the ratio of pseudo-curvatures of $\alpha = \alpha(s)$ is zero in terms of κ_ξ in $(26)_1$, we have κ_ξ which is undefined. Therefore, ξ is not an isotropic cubic.

Corollary 3.8. *Let $\xi = \xi(s_\xi)$ be the $\widetilde{E}_3$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is constant, then the $\widetilde{E}_3$ spherical indicatrix $\xi = \xi(s_\xi)$ is a pseudo-helix in $\mathbb{C}^3$.*

Proof. Let $\xi = \xi(s_\xi)$ be the $\widetilde{E}_3$ spherical image of a regular isotropic curve $\alpha = \alpha(s)$. If the pseudo-curvature of $\alpha = \alpha(s)$ is constant in terms of κ_ξ in $(26)_1$, we have $\kappa_\xi =$ constant. Therefore, ξ is a pseudo helix.

Theorem 3.6. *Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. The third component of the position vector of the curve with respect to pseudo arc-lengthed binormal spherical indicatrix satisfies a second order differential equation*

$$\frac{d^2 e_2}{ds_\xi^2} - (2 + \frac{1}{i}\frac{d\kappa}{ds_\xi})e_2 - \frac{1}{i}\frac{d\kappa}{ds_\xi}(\int e_2 ds_\xi) = 0. \tag{28}$$

Proof. Let $\vec{x} = \vec{x}(s)$ be an isotropic curve in $\mathbb{C}^3$. We know that $\frac{ds_\xi}{ds} = -i\kappa$ from $(25)_2$. Differentiating Cartan derivative equation $(5)_1$ with respect to pseudo arc-lengthed parameter of binormal spherical image, we obtain

$$\frac{de_1}{ds_\xi} = \frac{de_1}{ds}\frac{ds}{ds_\xi} = (-ie_2)(-\frac{1}{i\kappa}). \tag{29}$$

Rearranging (29), we have

$$\frac{de_1}{ds_\xi} = \frac{e_2}{\kappa}. \tag{30}$$

Similarly, if we differentiate $(5)_2$ and $(5)_3$, we obtain

$$\begin{aligned} \frac{de_2}{ds_\xi} &= \frac{de_2}{ds}\frac{ds}{ds_\xi} = -e_1 - \frac{1}{\kappa}e_3, \\ \frac{de_3}{ds_\xi} &= \frac{de_3}{ds}\frac{ds}{ds_\xi} = e_2. \end{aligned} \tag{31}$$

Thus differentiating Cartan derivative formulas with respect to pseudo-arc lengthed parameter of binormal spherical image, and using (30) and (31), we obtain

$$\begin{aligned} e_1' &= \frac{1}{\kappa}e_2, \\ e_2' &= -e_1 - \frac{1}{\kappa}e_3, \\ e_3' &= e_2. \end{aligned} \tag{32}$$

From (32), we have

$$e_3 = -[e_2' + e_1]\kappa. \tag{33}$$

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By derivatives of (33) and $(32)_1$, we obtain the equation (28). We characterized the isotropic curve $\vec{x} = \vec{x}(s)$ using the differential equation (28). If $\kappa = 0$, then the differential equation (28) becomes isotropic cubic. In this case, we get the differential equation as follows:

$$\frac{d^2 e_2}{ds_\xi^2} - 2e_2 = 0. \tag{34}$$

The solution of the differential equation (34) is obtained as follows:

$$e_2(s_\xi) = \mu_1 e^{-\sqrt{2}s_\xi} + \mu_2 e^{\sqrt{2}s_\xi}. \tag{35}$$

If $\kappa =$ constant, then the differential equation (28) characterizes isotropic helix. In this case, we obtain the differential equation as similar to (34), and so its solution is similar to (35).

Theorem 3.7. *Let $\alpha = \alpha(s)$ be an isotropic curve and all of $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$ be its spherical indicatrices in $\mathbb{C}^3$. Both of $\widetilde{E}_1$ and $\widetilde{E}_3$ are also spherical involutes of the $\widetilde{E}_3$ spherical indicatrix of α.*

Proof. Let us denote the isotropic tangent vectors of the spherical indicatrices $\widetilde{E}_1$, $\widetilde{E}_2$ and $\widetilde{E}_3$ as $e_{1\varphi}$, $e_{1\gamma}$, $e_{1\xi}$, respectively. By $(7)_1$, $(18)_1$, and $(25)_1$, these tangent vectors are given as

$$e_{1\varphi} = -e_2, \quad e_{1\gamma} = \sqrt{\frac{\kappa}{2}}e_1 + \frac{1}{\sqrt{2\kappa}}e_3, \quad e_{1\xi} = -e_2. \tag{36}$$

Using the equations (36), we have

$$\langle e_{1\varphi}, e_{1\gamma}\rangle = \langle e_{1\gamma}, e_{1\xi}\rangle = 0.$$

The tangent vectors of the $\widetilde{E}_1$ and $\widetilde{E}_3$ spherical images are orthogonal to tangent vector of the $\widetilde{E}_2$ spherical indicatrix, so the proof is completed.

3.2. Isotropic Darboux spherical indicatrices in $\mathbb{C}^3$

Definition 3.4. Let $\widetilde{w}_0 = \kappa e_1 - e_3$ be an isotropic Darboux vector. By translating the unit vector field $\widetilde{w}_0$ to the center O of the unit isotropic sphere S^2, we obtain an isotropic (pseudo) spherical image of $\xi = \xi(s_\xi)$ which is called Darboux spherical indicatrix in $\mathbb{C}^3$.

Theorem 3.8. *Let $\xi = \xi(s_\xi)$ be unit isotropic Darboux spherical indicatrix of the isotropic curve α in $\mathbb{C}^3$. The E. Cartan apparatus of $\widetilde{w}_0$ ($\{e_{1w}, e_{2w}, e_{3w}, \kappa_w\}$) can be formed according to E. Cartan apparatus of α ($\{e_1, e_2, e_3, \kappa\}$).*

Proof. Given the Darboux vector as follows

$$\widetilde{w}_0 = \widetilde{w}_0(s^*) = \kappa e_1 - e_3, \tag{37}$$

where s^* is the pseudo arc-parameter of $\widetilde{w}_0$. Differentiating (37) with respect to s, we get

$$\frac{d\widetilde{w}_0}{ds} = \frac{d\widetilde{w}_0}{ds^*}\frac{ds^*}{ds} = \kappa' e_1.$$

Therefore, we obtain

$$e_{1w} = e_1, \quad \frac{ds^*}{ds} = \kappa'. \tag{38}$$

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Using the definition of pseudo-curvature, we have the curvature κ_w of the curve $\widetilde{w}_0 = \widetilde{w}_0(s^*)$ as follows:

$$\kappa_w = 2(\frac{i}{\kappa})'\kappa + \frac{4}{(i\kappa')^2}(\frac{\kappa}{i})(\frac{\kappa}{i})'.$$

Using equation (3)$_2$, we obtain the principal normal as follows:

$$e_{2w} = \frac{i}{\kappa}(\kappa e_1 + e_3).$$

Finally, the binormal vector is as follows

$$e_{3w} = \frac{1}{i\kappa'}[(\frac{i}{\kappa})'\kappa e_1 + 2e_2 + (\frac{i}{\kappa})'e_3] - \kappa_w e_1.$$

Definition 3.5. An arbitrary isotropic curve $\vec{x} = \vec{x}(s)$ is a called an isotropic Darboux curve if it satisfies

$$\langle e_3, w\rangle = const.$$

for a non-zero constant w.

Theorem 3.9. *Let $\vec{x} = \vec{x}(s)$ be a Darboux helix, then the axis of the isotropic Darboux helix is as follows:*

$$d = \mp ie_1 - \frac{i}{\kappa}e_2 + \kappa e_3. \tag{39}$$

Proof. Let w be the vector field such that the function $\langle e_3, w\rangle = \kappa$ is constant. There exist $l_1(s)$ and $l_2(s)$ such that

$$d = l_1(s)e_1 + l_2(s)e_2 + \kappa e_3. \tag{40}$$

Differentiating (40) and using the derivative formulas in (5), we have

$$d' = (l_1' + l_2 i\kappa)e_1 + (l_2' + l_1 i)e_2 + (\kappa + l_2 i)e_3.$$

Since the system $\{e_1, e_2, e_3\}$ is linearly independent, we obtain

$$\begin{aligned} &l_1' + l_2 i\kappa = 0, \\ &l_2' + l_1 i = 0, \\ &\kappa + l_2 i = 0. \end{aligned} \tag{41}$$

From (41), we find

$$l_1 = \mp i, \quad l_2 = -\frac{i}{\kappa}. \tag{42}$$

Substituting (42) into (40), we get the axis of Darboux helix α as in (39).

Theorem 3.10. *Let $\vec{x} = \vec{x}(s)$ be an isotropic curve and a Darboux helix in $\mathbb{C}^3$. The pseudo curvature κ of the curve $\vec{x}(s)$ satisfies the following non-linear system of equations*

$$\mp 1 - (\frac{1}{\kappa})' - \kappa^2 i = 0, \quad \frac{1}{\kappa} + \kappa' = 0. \tag{43}$$

Proof. Since $\vec{x} = \vec{x}(s)$ is a Darboux helix, its axis is as in (39). Differentiating (39) gives us

$$d' = e_1 + [\mp 1 - (\frac{i}{\kappa})' - \kappa^2 i]e_2 + [\frac{1}{\kappa} + \kappa']e_3.$$

Since the system $\{e_2, e_3\}$ is linearly independent, we can write the system of equations (43).

Corollary 3.9. *Let $\vec{x} = \vec{x}(s)$ be an arbitrary isotropic curve. If $\vec{x} = \vec{x}(s)$ is an isotropic slant helix, then $w.d = constant$.*

Proof. It is known that the curve $\vec{x} = \vec{x}(s)$ is a pseudo-helix if and only if its pseudo-curvature κ is constant. From the inner product of w and d, we get

$$w.d = \kappa^2 \mp i.$$

Since $\kappa = \text{constant}$, we find out $w.d = \text{constant}$.

3.3. Spherical indicatrices of isotropic curves as Bertrand mates in $\mathbb{C}^3$

Theorem 3.11. *Let $\vec{x} = \vec{x}(s)$ be an isotropic curve in $\mathbb{C}^3$. Both of the $\widetilde{E}_1$ and $\widetilde{E}_3$ spherical indicatrices are Bertrand mates.*

Proof. Let us denote the $\widetilde{E}_1$ and $\widetilde{E}_3$ vectors as $e_{2\varphi}$ and $e_{2\xi}$, respectively. By the principal normal isotropic vectors in $(8)_1$ and $(26)_1$, we have the following equation

$$e_{2\varphi} = \frac{1}{\kappa} e_{2\xi},$$

so the $\widetilde{E}_1$ and $\widetilde{E}_3$ isotropic vectors are linearly dependent. As a result of that, they are Bertrand mates.

Theorem 3.12. *Let $\vec{x} = \vec{x}(s)$ be an isotropic curve in $\mathbb{C}^3$. If the $\widetilde{E}_1$ and $\widetilde{E}_3$ spherical indicatrices of isotropic curve are Bertrand mates, then*

$$\lambda(s) = constant.$$

Proof. From Definition 2.1, we write that

$$\widetilde{E}_1 = \widetilde{E}_3 + \lambda e_{2\xi}. \tag{44}$$

After derivation of (44) with respect to s, we obtain

$$\frac{d\widetilde{E}_1}{ds_\varphi}\frac{ds_\varphi}{ds} = \frac{d\widetilde{E}_3}{ds_\xi}\frac{ds_\xi}{ds} + \lambda' e_{2\xi} + \lambda e'_{2\xi}. \tag{45}$$

Rearranging (45) gives us

$$e_{1\varphi} i = -e_{1\xi} i\kappa + \lambda' e_{2\xi} - \lambda i(\kappa' e_1 - \kappa e_2 - i\kappa e_2). \tag{46}$$

From Definition 2.1, it yields that

$$e_{1\varphi} \perp e_{2\xi}.$$

Let's take the inner product (46) with $e_{2\xi}$, then we have

$$\frac{d\lambda}{ds} = 0 \tag{47}$$

which implies that $\lambda(s) = \text{constant}$.

3.4. Isotropic slant helices in $\mathbb{C}^3$

Definition 3.6. An arbitrary isotropic curve $\alpha = \alpha(s)$ is called a type-1 isotropic slant helix if it satisfies

$$e_2.u = \text{constant}, \tag{48}$$

for constant and non-zero u. An arbitrary isotropic curve $\alpha = \alpha(s)$ is called a type-2 isotropic slant helix if it satisfies

$$e_3.u = \text{constant}, \tag{49}$$

for constant and non-zero u.

Theorem 3.13. *Let $\alpha = \alpha(s)$ be an isotropic cubic in $\mathbb{C}^3$. If $\alpha = \alpha(s)$ is a type-1 isotropic slant helix, then the axis of the curve can be written as*

$$u = ce_1, \ c \textit{ being constant},$$

for $c \in \mathbb{C}^3$.

Proof. From Definitions 2.1 and 3.6, we know that $e_2.u = \text{constant}$, and $\kappa = 0$. Differentiating (48) with respect to pseudo-arc length parameter s, we find

$$i(e_3 + \kappa e_1).u = 0. \tag{50}$$

From (50), it is seen that $e_3 \perp u$, where

$$u = u_1 e_1 + u_2 e_2. \tag{51}$$

On the other hand, we see that u_2 is constant since

$$e_2.u = \text{constant}.$$

Differentiating (51) and using $u' = 0$, we find $u_1 = c$.

Theorem 3.14. *Let $\alpha = \alpha(s)$ be an isotropic curve in $\mathbb{C}^3$. Then $\alpha = \alpha(s)$ is a type-2 isotropic slant helix if and only if $\alpha = \alpha(s)$ is an isotropic cubic.*

Proof. From Definition 3.6, we know that $e_3.u = \text{constant}$, and $\kappa = 0$.
Differentiating (49), we find

$$i(e_3 + \kappa e_1).u = 0. \tag{52}$$

From (52), it is obvious that $e_2 \perp u$, where

$$u = v_1 e_1 + v_2 e_2. \tag{53}$$

Differentiating (53) and using $u' = 0$, we have

$$u' = (v_1' + v_2 i\kappa)e_1 - v_1 i e_2 + (v_2' + v_2 i)e_3 = 0.$$

Hence, we can write

$$v_1' + v_2 i\kappa = 0, \quad v_1 i = 0, \quad v_2' + v_2 i = 0. \tag{54}$$

From (54), it is seen that $\kappa = 0$; therefore, $\alpha(s)$ is an isotropic cubic.

4. Isotropic curves of constant breadth in $\mathbb{C}^3$

In this section, we define isotropic curves of constant breadth in $\mathbb{C}^3$, and we give some characterizations of these kind of curves.

Definition 4.1. A regular curve with more than 2-breadths in $\mathbb{C}^3$ is called an isotropic Smarandache breadth curve.

Let $\psi = \psi(s)$ be an isotropic Smarandache breadth curve. Moreover, let us suppose that $\psi = \psi(s)$ is a simple closed isotropic curve in $\mathbb{C}^3$. This curve will be denoted by (δ). The normal plane at every point P on the curve is also at a single point Q other than P. We call the point Q as the opposite point of P. We consider a curve in the class Γ as having parallel tangents T and T^* in opposite directions at the opposite points ψ and ψ^* of the curve. A simple closed curve having parallel tangents in opposite directions at opposite points can be represented with respect to Cartan frame by the equation

$$\psi^*(s) = \psi(s) + m_1 e_1 + m_2 e_2 + m_3 e_3, \tag{55}$$

where $m_i(s)$, $1 \leq i \leq 3$ are arbitrary functions, ψ and ψ^* are opposite points.

Differentiating (55) and considering Cartan equations, we have

$$\frac{d\psi^*}{ds} = \frac{d\psi^*}{ds^*}\frac{ds^*}{ds} = e_1^*\frac{ds^*}{ds} = (\frac{dm_1}{ds} + 1 + m_2 i\kappa)e_1 + (-m_1 i + \frac{dm_2}{ds})e_2 + (m_2 i + \frac{dm_3}{ds} - m_3 i\kappa)e_3. \tag{56}$$

Using $e_1^* = -e_1$, and rewriting (56) we obtain

$$\begin{cases} \dfrac{dm_1}{ds} = -m_2 i\kappa - \dfrac{ds^*}{ds} - 1, \\ \dfrac{dm_2}{ds} = m_1 i, \\ \dfrac{dm_3}{ds} = -m_2 i + m_3 i\kappa. \end{cases} \tag{57}$$

If we recall $g(s) = -\dfrac{ds^*}{ds} - 1$ and use it in (57), we get

$$\begin{cases} \dfrac{dm_1}{ds} = -m_2 i\kappa + g(s), \\ \dfrac{dm_2}{ds} = m_1 i, \\ \dfrac{dm_3}{ds} = -m_2 i + m_3 i\kappa. \end{cases} \tag{58}$$

Theorem 4.1. *Let $\overrightarrow{x} = \overrightarrow{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. The first component of the position vector of the curve with respect to E. Cartan frame satisfies a second order differential equation*

$$\frac{d^2 m_1(s)}{ds^2} + \frac{d}{ds}[\int m_1(s)ds.\kappa] - \frac{dg(s)}{ds} = 0. \tag{59}$$

Proof. From (58)$_1$, we obtain

$$m_2(s) = \frac{g(s) - \dfrac{dm_1(s)}{ds}}{i\kappa(s)}. \tag{60}$$

By substituting (60) into (58)$_2$, we reach the differential equation (59).

Now, we characterized the distance of constant breadth of the curve in $\mathbb{C}^3$. If the distance between opposite points of (δ) and (δ^*) is constant, then we can write that

$$\|\psi^* - \psi\| = m_2^2 + 2m_1 m_3 = l^2 = \text{constant}. \tag{61}$$

Hence, differentiating (61), we get

$$m_2 \frac{dm_2}{ds} + m_3 \frac{dm_1}{ds} + m_1 \frac{dm_3}{ds} = 0. \tag{62}$$

Let us study some cases for the special solution of (60) as follows:

From (61), we write that

$$m_3(m_1 i\kappa + \frac{dm_1}{ds}) = 0. \tag{63}$$

Obviously, there are two cases for (63) as follows

$$m_3 = 0 \quad \text{or} \quad \frac{dm_1}{ds} = -m_1 i\kappa. \tag{64}$$

Thus we shall study the following subcases of (64):

Case 1. If $m_3 = 0$, then $m_1 = m_2 = 0$. Hence the equation (55) becomes as

$$\psi^* = \psi.$$

Case 2. If $\dfrac{dm_1}{ds} = -m_1 i\kappa$, then we obtain

$$m_1 = -\int m_1 i\kappa ds,$$

$$m_2 = -\int (\int m_1 i\kappa ds) ds,$$

$$m_3 = e^{-i\kappa s}[\int e^{-i\kappa s}(-m_2 i)ds + l_4].$$

Hence the equation (55) becomes as

$$\psi^*(s) = \psi(s) + (-\int m_1 i\kappa ds)e_1 + (-\int (\int m_1 i\kappa ds)ds)e_2 + (e^{-i\kappa s}[\int e^{-i\kappa s}(-m_2 i)ds + l_4])e_3.$$

Theorem 4.2. *Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. There is an isotropic curve of constant breadth which lies fully in the subspace e_1.*

Proof. Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. Using $m_2 = m_3 = 0$ by means of (58), we obtain

$$m_1 = c_1 = \text{constant} \quad \text{and} \quad ds^* = -ds.$$

Therefore by (55), isotropic curve of constant breadth lies fully in the subspace e_1.

Theorem 4.3. *Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. There is an isotropic curve of constant breadth which lies fully in the subspace e_2.*

Proof. Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. If the components in $(58)_1$ and $(58)_3$ are taken as $m_1 = m_3 = 0$, it follows that

$$m_2 = c_2 = \text{constant} \quad \text{and} \quad \frac{ds^*}{ds} = -(c_2 i\kappa + 1).$$

Therefore, isotropic curve of constant breadth lies fully in the subspace e_2 by (55).

Theorem 4.4. *Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. There is an isotropic curve of constant breadth which lies fully in the subspace e_3.*

Proof. Let $\vec{x} = \vec{x}(s)$ be pseudo arc-lengthed isotropic curve in $\mathbb{C}^3$. If the components in $(58)_1$ and $(58)_3$ are taken as $m_1 = m_2 = 0$. It follows that

$$m_3 = e^{i\int \kappa ds} \quad \text{and} \quad ds^* = -ds.$$

Therefore by (56), isotropic curve of constant breadth lies fully in the subspace e_3.

The differential equation in (59) is also a characterization of the isotropic curve $\vec{x} = \vec{x}(s)$. The position vector of an arbitrary isotropic curve with respect to E. Cartan frame can be determined by means of its solution, however, a general solution of (59) has not been found yet. Because of this, let us suppose that $\vec{x}(s)$ is pseudo-helix for an explicit result. By this way, one can express

Corollary 4.1. *Let $\vec{x} = \vec{x}(s)$ be an isotropic (pseudo) helix in $\mathbb{C}^3$. In the case $\kappa =$ constant, position vector of $\vec{x}(s)$ with respect to E. Cartan frame can be written as*

$$\begin{aligned}\vec{x}(s) = \vec{x}^*(s) &+ \{l_1 e^{-\sqrt{\kappa}s} + l_2 e^{\sqrt{\kappa}s} + e^{-\sqrt{\kappa}s}[\int \frac{-g'(s)}{\sqrt{\kappa}e^{-\sqrt{\kappa}s}}ds] \\ &+ e^{\sqrt{\kappa}s}[\int \frac{g'(s)}{\sqrt{\kappa}e^{\sqrt{\kappa}s}}ds]\}e_1 + \{l_1 e^{-\sqrt{\kappa}s} + l_2 e^{\sqrt{\kappa}s} + e^{-\sqrt{\kappa}s}[\int \frac{-g'(s)}{\sqrt{\kappa}e^{-\sqrt{\kappa}s}}ds] \\ &+ e^{\sqrt{\kappa}s}[\int \frac{g'(s)}{\sqrt{\kappa}e^{\sqrt{\kappa}s}}ds]\}e_2 + \{e^{-i\kappa s}\int e^{-i\kappa s}(-m_2 i)ds + l_3\}e_3.\end{aligned}$$

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