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ON THE MATRIX NEARNESS PROBLEM FOR (SKEW–)SYMMETRIC MATRICES ASSOCIATED WITH THE MATRIX EQUATIONS $(A_1XB_1,\dots,A_kXB_k) = (C_1,\dots,C_k)$

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Abstract. Suppose that the matrix equations system $(A_1XB_1,\dots,A_kXB_k) = (C_1,\dots,C_k)$ with unknown matrix X is given, where A_i, B_i, and C_i, $i = 1,2,\dots,k$, are known matrices of suitable sizes. The matrix nearness problem is considered over the general and least squares solutions of this system. The explicit forms of the best approximate solutions of the problems over the sets of symmetric and skew–symmetric matrices are established as well. Moreover, a comparative table depending on some numerical examples in the literature is given.

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1. Introduction and Notations

Let $\mathcal{R}_{m,n}$, $\mathcal{R}_n$, $\mathcal{R}_n^S$, and $\mathcal{R}_n^{SS}$ be the sets of $m\times n$ real matrices, $n\times n$ real matrices, $n\times n$ real symmetric matrices, and $n\times n$ real skew–symmetric matrices, respectively. The symbols A^T and $A^\dagger$ will denote the transpose and the Moore–Penrose generalized inverse of a matrix $A\in\mathcal{R}_{m,n}$, respectively. Further, $vec(\cdot)$ will stand for the vec operator, i.e., $vec(A) = \left(a_1^T, a_2^T,\dots,a_n^T\right)^T$ for the matrix $A = (a_1,a_2,\dots,a_n)\in\mathcal{R}_{m,n}$, $a_i\in\mathcal{R}_{m,1}$, $i = 1,2,\dots,n$, and $A\otimes B$ will stand for the Kronecker product of matrices $A\in\mathcal{R}_{m,n}$ and $B\in\mathcal{R}_{p,r}$ (see [1]).

Moreover, let

$$\mathcal{R}_{m_1,n_1}\times\cdots\times\mathcal{R}_{m_k,n_k} = \{[A_1,\dots,A_k]\mid A_i\in\mathcal{R}_{m_i,n_i}, i = 1,2,\dots,k\}.$$

It is easy to see that $\mathcal{R}_{m_1,n_1}\times\cdots\times\mathcal{R}_{m_k,n_k}$ is a linear space over the real number field. We define the inner product for all $[A_1,A_2,\dots,A_k]$ and $[B_1,B_2,\dots,B_k]\in\mathcal{R}_{m_1,n_1}\times\mathcal{R}_{m_2,n_2}\times\cdots\times\mathcal{R}_{m_k,n_k}$ in the linear space as follows:

$$\langle[A_1,A_2,\dots,A_k],[B_1,B_2,\dots,B_k]\rangle = tr(B_1^TA_1)+\cdots+tr(B_k^TA_k),$$

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then $\mathcal{R}_{m_1,n_1} \times \mathcal{R}_{m_2,n_2} \times \cdots \times \mathcal{R}_{m_k,n_k}$ is a Hilbert inner space. Furthermore, let $\|\cdot\|_H$ denotes the norm that is derived by inner product, i.e.,

$$\begin{aligned} \|[A_1, A_2, \ldots, A_k]\|_H &= \langle [A_1, A_2, \ldots, A_k], [A_1, A_2, \ldots, A_k] \rangle^{1/2} \\ &= [tr(A_1^T A_1) + tr(A_2^T A_2) + \cdots + tr(A_k^T A_k)]^{1/2} \\ &= \left(\|A_1\|^2 + \ldots + \|A_k\|^2 \right)^{1/2}, \end{aligned}$$

where $\|\cdot\|$ denotes the Frobenius norm (see, for example, [11]).

The well–known linear matrix equation $AXB = C$, where A, B, C are known matrices of suitable sizes and X is the matrix of unknowns, were studied in the case of special solution structures, e.g. symmetric, triangular, Hermitian, nonnegative definite, reflexive, diagonal etc. using matrix decomposition such as the singular value decomposition (SVD), the generalized SVD (GSVD), the quotient SVD, and the canonical correlation decomposition (CCD) in [5, 6, 12, 15–17, 23, 36].

Now, first, consider the following two problems.

Problem 1. *For given matrices $A \in \mathcal{R}_{m,n}$, $B \in \mathcal{R}_{n,r}$, and $C \in \mathcal{R}_{m,r}$, find $\hat{X} \in \Omega$ such that*

$$\left\| A\hat{X}B - C \right\| = \min_{X \in \Omega} \|AXB - C\|,$$

where Ω is anyone of the sets of special matrices such as symmetric, skew–symmetric, Hermitian, reflexive etc.

Problem 2. *Let S_{E_1} be the solution set of Problem 1. For a given matrix $X_0 \in \mathcal{R}_n$, find $\hat{X} \in S_{E_1}$ such that*

$$\left\| \hat{X} - X_0 \right\| = \min_{X \in S_{E_1}} \|X - X_0\|.$$

Problem 2 which is very important in applied sciences is known as the matrix nearness problem in the literature and it has been extensively studied in recent years. For instance, Peng et al. [28] and Huang et al. [14] presented matrix iteration methods for finding the symmetric and skew–symmetric solutions of Problem 2, respectively. Peng et al. [27] gave the necessary and sufficient conditions for solvability of Problem 2 over reflexive and anti–reflexive matrices. Moreover, they obtained the explicit expression of the optimal approximation solution of Problem 2 when X is a reflexive and an anti–reflexive matrix. In these literatures, the linear matrix equation $AXB = C$ is consistent. But, it is rarely possible to satisfy the consistency condition of the linear matrix equation $AXB = C$, since the matrices A, B, and C occurring in practice are usually obtained from an experiment. When Problem 2 is inconsistent, Qui et al. [32], Lei et al. [18], and Peng [31] established iterative methods over the (skew–)symmetric, and (skew–)symmetric P–commuting matrices. On the other hand, Liao et al. [20], Huang et al. [13] and Zhao et al. [38] derived an explicit expressions of the least squares solution to Problem 2 when X is (skew–)symmetric and (P,Q)–orthogonal symmetric matrix, respectively. Moreover, Sarduvan et al. established the explicit forms of the best approximate solutions of Problem 2 when X is (P,Q)–orthogonal (skew–)symmetric matrix [33].

And now, consider the following two problems.

Problem 3. *For given matrices $A_1 \in \mathcal{R}_{m_1,n}$, $B_1 \in \mathcal{R}_{n,p_1}$, $C_1 \in \mathcal{R}_{m_1,p_1}$, $A_2 \in \mathcal{R}_{m_2,n}$, $B_2 \in \mathcal{R}_{n,p_2}$, and $C_2 \in \mathcal{R}_{m_2,p_2}$, find $\hat{X} \in \Omega$ such that*

$$\left\| [A_1\hat{X}B_1 - C_1, A_2\hat{X}B_2 - C_2] \right\|_H = \min_{X\in\Omega} \|[A_1XB_1 - C_1, A_2XB_2 - C_2]\|_H$$

where Ω is anyone of the sets of special matrices such as symmetric, skew–symmetric, Hermitian, reflexive etc.

Problem 4. *Let S_{E_2} be the solution set of Problem 3. For a given matrix $X_0 \in \mathcal{R}_n$, find $\hat{X} \in S_{E_2}$ such that*

$$\left\| \hat{X} - X_0 \right\| = \min_{X\in S_{E_2}} \|X - X_0\|.$$

Research on solving a pair of matrix equations has been actively ongoing for past years. For instance, Mitra [24] and Navarra [25] established conditions for the existence of a solution and a representation of a general common solution to Problem 3. Also, Özgüler et al. [26], Woude [35], Wang [37], and Liu [22] derived necessary and sufficient conditions for existence of common solution to Problem 3. Moreover, Dehghan et al. [8] obtained conditions for the existence of (R,S)–(skew–)symmetric solution of Problem 3. Ding et al. [9] presented an iterative method for solving a pair of inconsistent matrix equations.

Besides the works on finding the conditions for the existence of common solution to Problem 3, there are some valuable efforts on solving of the matrix nearness problem for a pair of matrix equations. For example, in the case that the matrix equations in Problem 3 are consistent, iterative algorithms were presented for solving Problem 4 with certain constraints on solution such as symmetric, reflexive, bisymmetric, generalized centro–symmetric, and generalized reflexive matrices in [3,7,29,30,34]. Cai et al. [2] and Chen et al. [4] derived iterative algorithms over bisymmetric and symmetric solutions, respectively, in the case that the matrix equations in Problem 4 are inconsistent.

It is noteworthy that when the pair of matrix equations is inconsistent, its least squares solutions with minimum norm cannot be obtained by GSVD and CCD. In order to over come this difficulty, Liao, Lei [19] and Liao et al. [21] derived a different approach based on the projection theorem. Therefore, they could used the method of GSVD and CCD to obtain the solution.

In this paper, it is established the general expressions of the (skew–)symmetric solutions to Problem 4 using kronecker product and Moore–Penrose inverse. Moreover, this general expressions are expanded for the matrix equations of the form

$(A_1 X B_1, \ldots, A_k X B_k) = (C_1, \ldots, C_k)$. Furthermore, a comparative table depending on some numerical examples in the literature is given.

2. Preliminary Results

The vector $x_0 \in \mathcal{R}_{n,1}$ is a least squares solution (LSS) to the inconsistent system of linear equations $Ax = g$, where $A \in \mathcal{R}_{m,n}$, if and only if

$$(Ax - g)^T (Ax - g) \geq (Ax_0 - g)^T (Ax_0 - g)$$

for all $x \in \mathcal{R}_{n,1}$ [10].

The vector $x_0 \in \mathcal{R}_{n,1}$ is the best approximate solution (BAS) to the inconsistent system of linear equations $Ax = g$, where $A \in \mathcal{R}_{m,n}$, if and only if

(1) $(Ax - g)^T (Ax - g) \geq (Ax_0 - g)^T (Ax_0 - g)$ for all $x \in \mathcal{R}_{n,1}$,
(2) $x^T x > x_0{}^T x_0$ for all $x \in \mathcal{R}_{n,1} \backslash \{x_0\}$ satisfying $(Ax - g)^T (Ax - g) = (Ax_0 - g)^T (Ax_0 - g)$ [10].

It is noteworthy that there may be many LSS for an inconsistent system of linear equations. In addition, a LSS may not be the BAS while the BAS is always a LSS. However, the BAS is always unique.

If it is assumed that the matrix equation $AXB = C$, where $A \in \mathcal{R}_{m,n}$, $B \in \mathcal{R}_{p,r}$, $C \in \mathcal{R}_{m,r}$ are known nonzero matrices and $X \in \mathcal{R}_{n,p}$ is the matrix of unknowns, is inconsistent, then it may be asked to find a matrix X such that $\|AXB - C\|$ is minimum. A matrix satisfying this condition is called an approximate solution to the matrix equation. The matrix $\hat{X} \in \mathcal{R}_{n,p}$ is defined to be the BAS to the matrix equation $AXB = C$ if and only if

(1) $\|AXB - C\| \geq \left\| A\hat{X}B - C \right\|$ for all $X \in \mathcal{R}_{n,p}$,
(2) $\|X\| > \left\| \hat{X} \right\|$ for all $X \in \mathcal{R}_{n,p} \backslash \left\{ \hat{X} \right\}$ satisfying $\|AXB - C\| = \left\| A\hat{X}B - C \right\|$.

We note that a vector $k \in \mathcal{R}_{mn,1}$ will stand for the vector $vec(K)$ in the rest of the text, where $K \in \mathcal{R}_{m,n}$.

It is known that the matrix equation $AXB = C$ can be equivalently written as

$$\left(B^T \otimes A \right) x = c. \tag{2.1}$$

Consequently, the solutions of a matrix equation $AXB = C$ can be obtained by considering the usual system of linear equations (2.1) instead of the matrix equation $AXB = C$. Now, we will give the following Lemma which can be proved easily.

Lemma 1 ([10]). *Suppose that S_g is the set of all solutions to the consistent system of linear equations $Ax = g$, where $A \in \mathcal{R}_{m,n}$ is a known matrix, $g \in \mathcal{R}_{m,1}$ is a known vector, and $x \in \mathcal{R}_{n,1}$ is the vector of unknowns. For a given vector $x_0 \in \mathcal{R}_{n,1}$, the vector $\hat{x} \in S_g$ satisfying*

$$\|\hat{x} - x_0\| = \min_{x \in S_g} \|x - x_0\|$$

is given by

$$\hat{x} = A^{\dagger}g + \left(I - A^{\dagger}A\right)x_0.$$

Lemma 2. *Let S_e be the set of all least squares solutions to the system of linear equations $Ax = g$ which do not need to be consistent, where $A \in \mathcal{R}_{m,n}$ is a known matrix, $g \in \mathcal{R}_{m,1}$ is a known vector, and $x \in \mathcal{R}_{n,1}$ is the vector of unknowns. For a given vector $x_0 \in \mathcal{R}_{n,1}$, the vector $\hat{x} \in S_e$ satisfying*

$$\|\hat{x} - x_0\| = \min_{x \in S_e} \|x - x_0\|$$

is given by

$$\hat{x} = A^{\dagger}g + \left(I - A^{\dagger}A\right)x_0.$$

Proof. If the system is consistent, then the proof is clear from Lemma 1. Now, let the system be inconsistent. Then, the normal equations of the system is

$$A^T Ax = A^T g \tag{2.2}$$

which is consistent. So, from Lemma 1, the BAS of the inconsistent system $Ax = g$ is

$$\hat{x} = (A^T A)^{\dagger} A^T g + \left(I - (A^T A)^{\dagger}(A^T A)\right) x_0$$

or, equivalently, from [10, Theorem 6.2.16]

$$\hat{x} = A^{\dagger}g + \left(I - A^{\dagger}A\right)x_0.$$

□

It is noteworthy that the structures of $\hat{x}$ in Lemmas 1 and 2 are exactly the same.

Remark 1. The minimization problem

$$\min \|X - X_0\|$$

is equivalent to the minimization problem

$$\min \left\| X - \frac{1}{2}\left(X_0 + X_0^T\right) \right\|$$

over the set of $\mathcal{R}_n^S$ since

$$\|X - X_0\|^2 = \left\| X - \frac{1}{2}\left(X_0 + X_0^T\right) \right\|^2 + \left\| \frac{1}{2}\left(X_0 - X_0^T\right) \right\|^2, \forall X \in \mathcal{R}_n^S.$$

So, the matrix $\frac{1}{2}\left(X_0 + X_0^T\right)$ instead of the matrix X_0 is taken to find the symmetric solutions of Problems 4 if the matrix X_0 is not symmetric.

Similarly if the matrix X_0 is not skew–symmetric, then the matrix $\frac{1}{2}\left(X_0 - X_0^T\right)$ instead of the matrix X_0 is taken to find the skew–symmetric solutions of Problem 4.

3. The (Skew–)Symmetric Solution of Problem 4

Our aim is to find a symmetric solution of Problem 4 with an arbitrary matrix $X_0 \in \mathcal{R}_n$. To do this, let us consider the quartet of matrix equations

$$\begin{aligned} A_1 X B_1 &= C_1 \\ B_1^T X A_1^T &= C_1^T \\ A_2 X B_2 &= C_2 \\ B_2^T X A_2^T &= C_2^T \end{aligned}$$

or, equivalently, the usual system of linear equations

$$\begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ B_2^T \otimes A_2 \\ A_2 \otimes B_2^T \end{bmatrix} x = \begin{bmatrix} vec\,(C_1) \\ vec\left(C_1^T\right) \\ vec\,(C_2) \\ vec\left(C_2^T\right) \end{bmatrix}. \tag{3.1}$$

In view of Lemma 2, the solution vector of the matrix nearness problem of the system (3.1) is

$$\hat{x} = \begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ B_2^T \otimes A_2 \\ A_2 \otimes B_2^T \end{bmatrix}^\dagger \begin{bmatrix} vec\,(C_1) \\ vec\left(C_1^T\right) \\ vec\,(C_2) \\ vec\left(C_2^T\right) \end{bmatrix} + x_0 - \begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ B_2^T \otimes A_2 \\ A_2 \otimes B_2^T \end{bmatrix}^\dagger \begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ B_2^T \otimes A_2 \\ A_2 \otimes B_2^T \end{bmatrix} x_0, \tag{3.2}$$

where $x_0 = vec(\frac{1}{2}(X_0 + X_0^T))$.

Thus, we have the following theorem within the framework of those.

Theorem 1. *Let $A_1 \in \mathcal{R}_{m_1,n}$, $B_1 \in \mathcal{R}_{n,p_1}$, $C_1 \in \mathcal{R}_{m_1,p_1}$, $A_2 \in \mathcal{R}_{m_2,n}$, $B_2 \in \mathcal{R}_{n,p_2}$, $C_2 \in \mathcal{R}_{m_2,p_2}$, $X_0 \in \mathcal{R}_n$ are known matrices, and $x_0 = vec(\frac{1}{2}(X_0 + X_0^T))$. Then the symmetric solution $\hat{X} \in \mathcal{R}_n^S$ of Problem 4 is given as in (3.2) in view of $\hat{x} = vec(\hat{X})$.*

If it is required to find skew–symmetric solution of Problem 4, then $vec\left(-C_i^T\right)$ is taken instead of $vec\left(C_i^T\right)$, $i = 1,2$.

By continuing with the same idea, Theorem 1 can be extended to k matrix equations where k is an arbitrary positive integer.

Theorem 2. *Let $A_i \in \mathcal{R}_{m_i,n}$, $B_i \in \mathcal{R}_{n,p_i}$, $C_i \in \mathcal{R}_{m_i,p_i}$, $i = 1,2,\ldots,k$, are known matrices and*

$$S_E = \{X \mid X \in \Omega, \|[A_1XB_1 - C_1, \ldots, A_kXB_k - C_k]\|_H = min\}. \tag{3.3}$$

For a given matrix $X_0 \in \mathcal{R}_n$ the symmetric solution $\hat{X} \in S_E$ satisfying

$$\left\|\hat{X} - X_0\right\| = \min_{X \in S_E} \|X - X_0\|$$

is given by

$$\hat{x} = \begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ \vdots \\ B_k^T \otimes A_k \\ A_k \otimes B_k^T \end{bmatrix}^{\dagger} \begin{bmatrix} vec\,(C_1) \\ vec\,(C_1^T) \\ \vdots \\ vec\,(C_k) \\ vec\,(C_k^T) \end{bmatrix} + x_0 - \begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ \vdots \\ B_k^T \otimes A_k \\ A_k \otimes B_k^T \end{bmatrix}^{\dagger} \begin{bmatrix} B_1^T \otimes A_1 \\ A_1 \otimes B_1^T \\ \vdots \\ B_k^T \otimes A_k \\ A_k \otimes B_k^T \end{bmatrix} x_0, \tag{3.4}$$

where $x_0 = vec(\frac{1}{2}(X_0 + X_0^T))$ *and,* $\hat{x} = vec(\hat{X})$.

Similarly, $vec\left(-C_i^T\right)$ is taken instead of $vec\left(C_i^T\right)$, $i = 1,2,\ldots,k$, for finding skew–symmetric solution.

TABLE 1. A comparative table for the examples chosen from the literature

<table>
<tr><th></th><th>ε (★)</th><th>$\|X - X_0\|$</th><th>$\|[A_1XB_1 - C_1, A_2XB_2 - C_2]\|_H$</th></tr>
<tr><td rowspan="8">Example 2 in [19]</td><td rowspan="2">1</td><td>43.6600</td><td>4.5687e+003</td></tr>
<tr><td>43.6600</td><td>4.5687e+003</td></tr>
<tr><td rowspan="2">0.01</td><td>0.4366</td><td>45.6873</td></tr>
<tr><td>0.4366</td><td>45.6873</td></tr>
<tr><td rowspan="2">0.0001</td><td>0.0044</td><td>0.4569</td></tr>
<tr><td>0.0044</td><td>0.4569</td></tr>
<tr><td rowspan="2">0.000001</td><td>4.3660e-005</td><td>0.0046</td></tr>
<tr><td>4.3660e-005</td><td>0.0046</td></tr>
<tr><td rowspan="8">Example 3 in [21]</td><td rowspan="2">1</td><td>28.7113</td><td>6.1988e+003</td></tr>
<tr><td>33.6729</td><td>4.5687e+003</td></tr>
<tr><td rowspan="2">0.01</td><td>0.2871</td><td>66.9884</td></tr>
<tr><td>0.3367</td><td>48.1408</td></tr>
<tr><td rowspan="2">0.0001</td><td>0.0029</td><td>0.6199</td></tr>
<tr><td>0.0034</td><td>0.4568</td></tr>
<tr><td rowspan="2">0.000001</td><td>2.8711e-005</td><td>0.0062</td></tr>
<tr><td>3.3673e-005</td><td>0.0046</td></tr>
<tr><td rowspan="2">Example 1 in [29]</td><td rowspan="2"></td><td>413.4852</td><td>2.6688e-021</td></tr>
<tr><td>413.4852</td><td>4.4970e-021</td></tr>
<tr><td rowspan="2">Example 4.1 in [34]</td><td rowspan="2"></td><td>69.9995</td><td>9.0727e-023</td></tr>
<tr><td>69.9995</td><td>1.3436e-021</td></tr>
</table>

(★) ε is as given in [19, 21].

We close this section with a comparative table (Table 1) consisting examples chosen from the literature. In each cell, the first value is the result obtained by the method proposed in this work while the second one is the result in the referenced work. All the computations have been performed using Matlab 7.5.

4. CONCLUSIONS

To solve matrix equations system problems become relatively difficult when it is used matrix decompositions. For example, if the matrix equations are inconsistent, the matrix decompositions GSVD and CCD can not be individually used to solve them, and the difficulty lies in the fact that the invariance of the Frobenius norm does not hold for general nonsingular matrices in these decompositions [19]. For this reason, these kinds of Problems are usually solved using iterative methods. However, it is a well known fact that solving these kinds of problems by elementary methods, which are very simple and elegant, eliminates errors caused by processes of iteration. Due to these kinds of facts, in our opinion, it is better to give the explicit analytical expressions of the solutions obtained by elementary methods instead of giving, especially, the implicit solutions obtained by iterative methods for inconsistent matrix equations encountered in most of physical problems.

If the dimensions and elements of the matrices included in the problems are large and sparse, it is clear that the computing processes, especially in the elementary methods, have contained highly large number of terms. Therefore, elementary methods may not be useful with the current computer technology in these kinds of situations. On the other hand, the speed of technological developments is incredible. So, we believe that these difficulties will be disappeared in the nearest future. Consequently, within the framework of these considerations, to establish the solutions as in this note are important not only mathematical point of view but also practically.

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