



Hermite-Hadamard Type Inequalities for Exponentially p -Convex Stochastic Processes

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Abstract

In this paper, the concept of exponentially p -convex stochastic process is introduced. Several new inequalities of Hermite-Hadamard type for exponentially p -convex stochastic process are established. Some special cases are given which are obtained from our main results. The results obtained in this work are the generalizations of the known results.

Keywords: Convex stochastic processes, p -convex stochastic processes, exponentially p -convex stochastic processes, mean-square integral, Hermite-Hadamard type inequality

1. INTRODUCTION

Stochastic process is a research area in probability theory dealing with probabilistic models that develop over time. It is seen as a branch of mathematics, because it starts with the axioms of probability and gives rise to remarkable results about those axioms. Even though those results are applicable to many areas, at first they are best understood with regard to their mathematical structures.

Stochastic convexity is of great importance in statistics and probability, also in optimization, because it provides numerical approximations when there exist probabilistic quantities.

In 1980, Nikodem [8] defined convex stochastic processes and investigated their properties. In 1988, Shaked et al. [13] defined stochastic

convexity and gave its applications. In 1992, Skowronski [14] introduced some new types of convex stochastic processes and obtained some further results on these processes. In 2012, Kotrys [3] extended classical Hermite-Hadamard inequality to convex stochastic processes.

In recent years, there have been many studies on the above mentioned processes. For recent generalizations and improvements on convex stochastic processes, please refer to [2]-[6], [9]-[12], [15], [16].

2. PRELIMINARIES

Suppose (Ω, ξ, P) be a probability space and $X: \Omega \rightarrow \mathbb{R}$ be a function. If the function X is ξ -measurable it is called a random variable. Suppose $I \subset \mathbb{R}$ be an interval. A function $X: I \times \Omega \rightarrow \mathbb{R}$ is called a stochastic process, if the function $X(s, \cdot)$ is a random variable for all $s \in I$.

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Let $P - \lim$ and $E[X(s, \cdot)]$ denote the limit in probability and the expectation value of random variable $X(t, \cdot)$, respectively. Then, a stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ has

(1) continuity in probability in I , if for every $s_0 \in I$

$$P - \lim_{s \rightarrow s_0} X(s, \cdot) = X(s_0, \cdot).$$

(2) mean-square continuity in I , if for all $s_0 \in I$

$$\lim_{s \rightarrow s_0} E \left[(X(s, \cdot) - X(s_0, \cdot))^2 \right] = 0.$$

(3) mean-square differentiability at a point $s \in I$ if there exists a random variable $X'(s, \cdot): I \times \Omega \rightarrow \mathbb{R}$ such that

$$X'(s, \cdot) = P - \lim_{s \rightarrow s_0} \frac{X(s, \cdot) - X(s_0, \cdot)}{s - s_0}.$$

Note that if the stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ has mean-square continuity, then it has continuity in probability, but the converse is not true.

Let $X: I \times \Omega \rightarrow \mathbb{R}$ be a stochastic process with $E \left[(X(s, \cdot))^2 \right] < \infty$ and for all $s \in I$. Let $c = s_0 < s_1 < s_2 < \dots < s_n = d$ be a partition of $[c, d]$, if the identity

$$\lim_{n \rightarrow \infty} E \left[(X(\theta_j)(s_j - s_{j-1}) - Y)^2 \right] = 0$$

holds for all normal sequences of partitions of $[c, d]$ and for all $\theta_j \in [s_{j-1}, s_j]$, $k = 1, 2, \dots, n$. Then, we can write

$$Y(\cdot) = \int_c^d X(t, \cdot) dt \quad (\text{almost everywhere})$$

The assumption of the mean-square continuity of the stochastic process X is enough for the mean-square integral to exist.

Definition 2.1. [8] The stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is said to be convex if for all $\theta \in [0, 1]$ and $c, d \in I$ the inequality

$$X(\theta c + (1 - \theta)d, \cdot) \leq \theta X(c, \cdot) + (1 - \theta)X(d, \cdot) \quad (1)$$

is satisfied almost everywhere. If the inequality (1) is assumed only for $\lambda = \frac{1}{2}$, then the stochastic process X is called Jensen-convex or $\frac{1}{2}$ -convex.

Theorem 2.2. [3] Let $X: I \times \Omega \rightarrow \mathbb{R}$ be a Jensen-convex stochastic process and mean-square continuous in I . Then for every $c, d \in I$, $c < d$, the inequality

$$X\left(\frac{c+d}{2}, \cdot\right) \leq \frac{1}{d-c} \int_c^d X(s, \cdot) ds \leq \frac{X(c, \cdot) + X(d, \cdot)}{2} \quad (2)$$

is satisfied almost everywhere.

Definition 2.3. [9] The stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is called a p -convex stochastic process if the inequality

$$X\left([\theta c^p + (1 - \theta)d^p]^{\frac{1}{p}}, \cdot\right) \leq \theta X(c, \cdot) + (1 - \theta)X(d, \cdot)$$

holds almost everywhere for all $c, d \in I \subset (0, \infty)$, $\theta \in [0, 1]$ and $p \in \mathbb{R} \setminus \{0\}$.

Theorem 2.4. [9] Let $X: I \subset (0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a p -convex stochastic process and mean-square integrable on $[c, d]$ where $c, d \in I$ and $c < d$. Then

$$X\left(\left[\frac{c^p + d^p}{2}\right]^{\frac{1}{p}}, \cdot\right) \leq \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \leq \frac{X(c, \cdot) + X(d, \cdot)}{2} \quad (3)$$

Lemma 2.5. [9] Let $X: I \subset (0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a mean-square differentiable stochastic process on I° (the interior of I) and $c, d \in I$, $c < d$ and $p \in \mathbb{R} \setminus \{0\}$. If X' is mean-square integrable on $[c, d]$, then the following equality holds almost everywhere:

$$\begin{aligned} & \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \\ &= \frac{d^p - c^p}{2p} \int_0^1 \frac{1 - 2\theta}{[\theta c^p + (1 - \theta)d^p]^{1-\frac{1}{p}}} \\ & \quad \times X'([\theta c^p + (1 - \theta)d^p]^{\frac{1}{p}}, \cdot) d\theta. \end{aligned}$$

Definition 2.6. [7] A function $f: I \subset (0, \infty) \rightarrow \mathbb{R}$ is called exponentially p -convex if the inequality

$$f\left([\lambda u^p + (1-\lambda)v^p]^{\frac{1}{p}}\right) \leq \lambda \frac{f(u)}{e^{\alpha u}} + (1-\lambda) \frac{f(v)}{e^{\alpha v}}$$

holds for all $u, v \in I$, $p \in \mathbb{R} \setminus \{0\}$, $\lambda \in [0,1]$ and $\alpha \in \mathbb{R}$.

Now we recall the following special functions (see [1]).

The beta function is defined as:

$$\beta(x, y) = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda, \quad x > 0, y > 0.$$

The hypergeometric function is as follows:

$$\begin{aligned} & {}_2F_1(a, b; c; z) \\ &= \frac{1}{\beta(b, c-b)} \int_0^1 \lambda^{b-1} (1-\lambda)^{c-b-1} (-z\lambda)^{-a} d\lambda \end{aligned}$$

for $c > b > 0, |z| < 1$.

3. MAIN RESULTS

In this section we introduce a new concept, which is called exponentially p -convex stochastic process. We establish new Hermite-Hadamard type inequalities for exponentially p -convex stochastic process. We also give some special cases obtained from our main results.

Definition 3.1. The stochastic process $X: I \times \Omega \rightarrow \mathbb{R}$ is called exponentially p -convex, if the following inequality holds almost everywhere:

$$\begin{aligned} X\left([\theta c^p + (1-\theta)d^p]^{\frac{1}{p}}, \cdot\right) \\ \leq \theta \frac{X(c, \cdot)}{e^{\alpha c}} + (1-\theta) \frac{X(d, \cdot)}{e^{\alpha d}} \end{aligned} \quad (4)$$

for all $c, d \in I \subset (0, \infty)$, $\theta \in [0,1]$, $p \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. If the inequality (4) is reversed, then the process X is called exponentially p -concave.

It can be easily seen that, an exponentially p -convex stochastic process reduces to p -convex and convex stochastic processes for $\alpha = 0$ and $(\alpha, p) = (0,1)$, respectively.

Theorem 3.2. Let $X: I \subset (0, \infty) \rightarrow \mathbb{R}$ be an exponentially p -convex stochastic process. Let $c, d \in I$ with $c < d$. If X is mean-square integrable on $[c, d]$, then for $p \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$, we have almost everywhere

$$\begin{aligned} X\left(\left[\frac{c^p + d^p}{2}\right]^{\frac{1}{p}}, \cdot\right) &\leq \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p} e^{\alpha s}} ds \\ &\leq A_1(\theta) \frac{X(c, \cdot)}{e^{\alpha c}} + A_2(\theta) \frac{X(d, \cdot)}{e^{\alpha d}} \end{aligned} \quad (5)$$

where

$$\begin{aligned} A_1(\theta) &= \int_0^1 \frac{\theta d\theta}{e^{\alpha(\theta c^p + (1-\theta)d^p)^{\frac{1}{p}}}}, \\ A_2(\theta) &= \int_0^1 \frac{(1-\theta)d\theta}{e^{\alpha(\lambda c^p + (1-\theta)d^p)^{\frac{1}{p}}}}. \end{aligned}$$

Proof. From exponential p -convexity of the stochastic process X , we have

$$2X\left(\left[\frac{s^p + t^p}{2}\right]^{\frac{1}{p}}, \cdot\right) \leq \frac{X(s, \cdot)}{e^{\alpha s}} + \frac{X(t, \cdot)}{e^{\alpha t}}.$$

Let $s^p = \theta c^p + (1-\theta)d^p$ and $t^p = (1-\theta)c^p + \theta d^p$. So, we get

$$\begin{aligned} 2X\left(\left[\frac{c^p + d^p}{2}\right]^{\frac{1}{p}}, \cdot\right) &\leq \frac{X([\theta c^p + (1-\theta)d^p]^{\frac{1}{p}}, \cdot)}{e^{\alpha(\theta c^p + (1-\theta)d^p)^{\frac{1}{p}}}} \\ &\quad + \frac{X([(1-\theta)c^p + \theta d^p]^{\frac{1}{p}}, \cdot)}{e^{\alpha((1-\theta)c^p + \theta d^p)^{\frac{1}{p}}}}. \end{aligned}$$

Integrating with respect to $\theta \in [0,1]$ and applying the change of variable method, we have

$$X\left(\left[\frac{c^p + d^p}{2}\right]^{\frac{1}{p}}, \cdot\right) \leq \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p} e^{\alpha s}} ds. \quad (6)$$

Thus, the left-hand side of the inequality (5) is established. For the right-hand side of the inequality (5), again utilizing the exponential p -convexity of the stochastic process X , we have

$$\frac{X\left([\theta c^p + (1-\theta)d^p]^{\frac{1}{p}}, \cdot\right)}{e^{\alpha(\theta c^p + (1-\theta)d^p)^{\frac{1}{p}}}} \leq \frac{\theta \frac{X(c, \cdot)}{e^{\alpha c}} + (1-\theta) \frac{X(d, \cdot)}{e^{\alpha d}}}{e^{\alpha(\theta c^p + (1-\theta)d^p)^{\frac{1}{p}}}}$$

Integrating with respect to θ on $[0,1]$, we have

$$\begin{aligned} \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p} e^{\alpha s}} ds &\leq \frac{X(c, \cdot)}{e^{\alpha c}} \int_0^1 \frac{\theta d\theta}{e^{\alpha(\theta c^p + (1-\theta)d^p)^{\frac{1}{p}}}} \\ &+ \frac{X(d, \cdot)}{e^{\alpha d}} \int_0^1 \frac{(1-\theta)d\theta}{e^{\alpha(\theta c^p + (1-\theta)d^p)^{\frac{1}{p}}}}. \end{aligned} \quad (7)$$

A combination of inequalities (6) and (7) gives inequality (5).

Remark 3.3. Choosing $\alpha = 0$ in Theorem 3.2, we get inequality (3) in Theorem 2.4.

Remark 3.4. By taking $(\alpha, p) = (0, 1)$ in Theorem 3.2, we attain inequality (2) in Theorem 2.2.

Theorem 3.5. Let $X: I \subset (0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a differentiable stochastic process on I° and X' be mean-square integrable on $[c, d]$. If $|X'|^q$ is exponentially p -convex stochastic process on $[c, d]$ for $q \geq 1$, $c, d \in I^\circ$, $c < d$ and $p \in \mathbb{R} \setminus \{0\}$, then the following inequality holds almost everywhere

$$\begin{aligned} &\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right| \\ &\leq \frac{d^p - c^p}{2p} B_1^{1-\frac{1}{q}} \left[B_2 \left| \frac{X'(c, \cdot)}{e^{\alpha c}} \right|^q + B_3 \left| \frac{X'(d, \cdot)}{e^{\alpha d}} \right|^q \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} B_1 &= B_1(c, d; p) = \frac{1}{4} \left(\frac{c^p + d^p}{2} \right)^{\frac{1}{p}-1} \\ &\times \left[{}_2F_1 \left(1 - \frac{1}{p}, 2; 3; \frac{c^p - d^p}{c^p + d^p} \right) + {}_2F_1 \left(1 - \frac{1}{p}, 2; 3; \frac{d^p - c^p}{c^p + d^p} \right) \right], \end{aligned}$$

$$\begin{aligned} B_2 &= B_2(c, d; p) = \frac{1}{24} \left(\frac{c^p + d^p}{2} \right)^{\frac{1}{p}-1} \\ &\times \left[{}_2F_1 \left(1 - \frac{1}{p}, 2; 4; \frac{c^p - d^p}{c^p + d^p} \right) + 6 {}_2F_1 \left(1 - \frac{1}{p}, 2; 3; \frac{d^p - c^p}{c^p + d^p} \right) \right. \\ &\quad \left. + {}_2F_1 \left(1 - \frac{1}{p}, 2; 4; \frac{d^p - c^p}{c^p + d^p} \right) \right] \end{aligned}$$

and

$$B_3 = B_3(c, d; p) = B_1 - B_2.$$

Proof. From Lemma 2.5 and using power-mean integral inequality, we have

$$\begin{aligned} &\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right| \\ &= \frac{d^p - c^p}{2p} \int_0^1 \frac{1-2\theta}{[\theta c^p + (1-\theta)d^p]^{1-\frac{1}{p}}} X'([\theta c^p + (1-\theta)d^p]^{\frac{1}{p}}, \cdot) d\theta \\ &\leq \frac{d^p - c^p}{2p} \int_0^1 \left| \frac{1-2\theta}{[\theta c^p + (1-\theta)d^p]^{1-\frac{1}{p}}} \right| \\ &\quad \times \left| X'([\theta c^p + (1-\theta)d^p]^{\frac{1}{p}}, \cdot) \right| d\theta \\ &\leq \frac{d^p - c^p}{2p} \left(\int_0^1 \frac{|1-2\theta|}{[\theta c^p + (1-\theta)d^p]^{1-\frac{1}{p}}} d\theta \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 \frac{|1-2\theta|}{[\theta c^p + (1-\theta)d^p]^{1-\frac{1}{p}}} \left| X'([\theta c^p + (1-\theta)d^p]^{\frac{1}{p}}, \cdot) \right|^q d\theta \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|X'|^q$ is exponentially p -convex stochastic process on $[c, d]$, we have almost everywhere

$$\begin{aligned} &\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right| \\ &\leq \frac{d^p - c^p}{2p} \left(\int_0^1 \frac{|1-2\theta|}{[\theta c^p + (1-\theta)d^p]^{1-\frac{1}{p}}} d\theta \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 \frac{|1-2\theta| \left[\theta \left| \frac{X'(c, \cdot)}{e^{\alpha c}} \right|^q + (1-\theta) \left| \frac{X'(d, \cdot)}{e^{\alpha d}} \right|^q \right]}{[\theta c^p + (1-\theta)d^p]^{1-\frac{1}{p}}} d\theta \right)^{\frac{1}{q}} \\ &\leq \frac{d^p - c^p}{2p} B_1^{1-\frac{1}{q}} \left[B_2 \left| \frac{X(c, \cdot)}{e^{\alpha c}} \right|^q + B_3 \left| \frac{X(d, \cdot)}{e^{\alpha d}} \right|^q \right]^{\frac{1}{q}}. \end{aligned}$$

where

$$\int_0^1 \frac{|1-2\theta|}{[\theta u^p + (1-\theta)v^p]^{1-\frac{1}{p}}} d\theta = B_1(c, d; p),$$

$$\int_0^1 \frac{|1 - 2\theta|\theta}{[\theta c^p + (1 - \theta)d^p]^{1-\frac{1}{p}}} d\theta = B_2(c, d; p),$$

$$\int_0^1 \frac{|1 - 2\theta|(1 - \theta)}{[\theta c^p + (1 - \theta)d^p]^{1-\frac{1}{p}}} d\theta = B_1(c, d; p) - B_2(c, d; p).$$

Thus, the proof is completed.

Remark 3.6. If we choose $\alpha = 0$ in Theorem 3.5, we attain Theorem 4 in [9].

Remark 3.7. By choosing $(\alpha, p) = (0, 1)$ in Theorem 3.5, we attain Theorem 5 in [9].

Corollary 3.8. Under the conditions of Theorem 3.5, if we take $q = 1$, then

$$\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right|$$

$$\leq \frac{d^p - c^p}{2p} \left[B_2 \left| \frac{X'(c, \cdot)}{e^{\alpha c}} \right| + B_3 \left| \frac{X'(d, \cdot)}{e^{\alpha d}} \right| \right] \quad (\text{a. e.}),$$

where B_2 and B_3 are given in Theorem 3.5.

Remark 3.9. If $\alpha = 0$ in Corollary 3.8, we attain Corollary 4 in [9].

Remark 3.10. By letting $(\alpha, p) = (0, 1)$ in Corollary 3.8, we attain Theorem 5 in [9].

Theorem 3.11. Let $X: I \subset (0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a differentiable stochastic process on I° and X' be mean-square integrable on $[c, d]$. If $|X'|^q$ is exponentially p -convex stochastic process on $[c, d]$ for $q, r > 1, \frac{1}{q} + \frac{1}{r} = 1$, then the following inequality holds almost everywhere

$$\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right|$$

$$\leq \frac{d^p - c^p}{2p} \left(\frac{1}{r+1} \right)^{\frac{1}{r}} \left[B_4 \left| \frac{X'(c, \cdot)}{e^{\alpha c}} \right|^q + B_5 \left| \frac{X'(d, \cdot)}{e^{\alpha d}} \right|^q \right]^{\frac{1}{q}},$$

where

$$B_4 = B_4(c, d; p; q)$$

$$= \begin{cases} \frac{1}{2c^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 1; 3; 1 - \left(\frac{d}{c} \right)^p \right), & p < 0, \\ \frac{1}{2d^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 2; 3; 1 - \left(\frac{c}{d} \right)^p \right), & p > 0, \end{cases}$$

$$B_5 = B_5(c, d; p; q)$$

$$= \begin{cases} \frac{1}{2c^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 2; 3; 1 - \left(\frac{d}{c} \right)^p \right), & p < 0, \\ \frac{1}{2d^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 1; 3; 1 - \left(\frac{c}{d} \right)^p \right), & p > 0. \end{cases}$$

Proof. Using Lemma 2.5, Hölder's integral inequality and exponential p -convexity of the stochastic process $|X'|^q$ on $[c, d]$, we have almost everywhere

$$\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right|$$

$$\leq \frac{d^p - c^p}{2p} \left(\int_0^1 |1 - 2\theta|^r d\theta \right)^{\frac{1}{r}}$$

$$\times \left(\int_0^1 \frac{1}{[\theta c^p + (1 - \theta)d^p]^{q(1-\frac{1}{p})}} \left| X' \left([\theta c^p + (1 - \theta)d^p]^{\frac{1}{p}}, \cdot \right) \right|^q d\theta \right)^{\frac{1}{q}}$$

$$\leq \frac{d^p - c^p}{2p} \left(\frac{1}{r+1} \right)^{\frac{1}{r}} \left(\frac{\theta \left| \frac{X'(c, \cdot)}{e^{\alpha c}} \right|^q + (1 - \theta) \left| \frac{X'(d, \cdot)}{e^{\alpha d}} \right|^q}{[\theta c^p + (1 - \theta)d^p]^{q(1-\frac{1}{p})}} d\theta \right)^{\frac{1}{q}}$$

$$\leq \frac{d^p - c^p}{2p} \left(\frac{1}{r+1} \right)^{\frac{1}{r}} \left[B_4 \left| \frac{X'(c, \cdot)}{e^{\alpha c}} \right|^q + B_5 \left| \frac{X'(d, \cdot)}{e^{\alpha d}} \right|^q \right]^{\frac{1}{q}},$$

where

$$B_4 = \int_0^1 \frac{\theta}{[\theta c^p + (1 - \theta)d^p]^{q(1-\frac{1}{p})}} d\theta$$

$$= \begin{cases} \frac{1}{2c^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 1; 3; 1 - \left(\frac{d}{c} \right)^p \right), & p < 0, \\ \frac{1}{2d^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 2; 3; 1 - \left(\frac{c}{d} \right)^p \right), & p > 0, \end{cases}$$

$$B_5 = \int_0^1 \frac{1 - \theta}{[\theta c^p + (1 - \theta)d^p]^{q(1-\frac{1}{p})}} d\theta$$

$$= \begin{cases} \frac{1}{2c^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 2; 3; 1 - \left(\frac{d}{c} \right)^p \right), & p < 0, \\ \frac{1}{2d^{qp-q}} {}_2F_1 \left(q - \frac{q}{p}, 1; 3; 1 - \left(\frac{c}{d} \right)^p \right), & p > 0. \end{cases}$$

Remark 3.12. If we take $\alpha = 0$ in Theorem 3.11, we attain Theorem 6 in [9].

Theorem 3.13. Let $X: I \subset (0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a differentiable stochastic process on I° and X' be mean-square integrable on $[c, d]$. If $|X'|^q$ is exponentially p -convex stochastic process on $[c, d]$ for $q, r > 1, \frac{1}{q} + \frac{1}{r} = 1$, then

$$\left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right| \leq \frac{d^p - c^p}{2p} B_6^{\frac{1}{q}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{|X'(c, \cdot)|^q + |X'(d, \cdot)|^q}{2} \right)^{\frac{1}{q}},$$

where

$$B_6 = B_6(c, d; p; r)$$

$$= \begin{cases} \frac{1}{2c^{rp-r}} {}_2F_1\left(r - \frac{r}{p}, 1; 2; 1 - \left(\frac{d}{c}\right)^p\right), & p < 0, \\ \frac{1}{2d^{rp-r}} {}_2F_1\left(r - \frac{r}{p}, 1; 2; 1 - \left(\frac{c}{d}\right)^p\right), & p > 0, \end{cases}$$

Proof. From Lemma 2.5, using Hölder’s integral inequality and exponential p -convexity of the stochastic process $|X'|^q$ on $[c, d]$, we have almost everywhere

$$\begin{aligned} & \left| \frac{X(c, \cdot) + X(d, \cdot)}{2} - \frac{p}{d^p - c^p} \int_c^d \frac{X(s, \cdot)}{s^{1-p}} ds \right| \\ &= \frac{d^p - c^p}{2p} \int_0^1 \frac{1 - 2\theta}{[\theta c^p + (1 - \theta)d^p]^{1-\frac{1}{p}}} X'([\theta c^p + (1 - \theta)d^p]^{\frac{1}{p}}, \cdot) d\theta \\ &\leq \frac{d^p - c^p}{2p} \left(\int_0^1 \frac{1}{[\theta c^p + (1 - \theta)d^p]^{r(1-\frac{1}{p})}} d\theta \right)^{\frac{1}{r}} \\ &\quad \times \left(\int_0^1 |1 - 2\theta|^q \left| X'([\theta c^p + (1 - \theta)d^p]^{\frac{1}{p}}, \cdot) \right|^q d\theta \right)^{\frac{1}{q}} \\ &\leq \frac{d^p - c^p}{2p} B_6^{\frac{1}{q}} \left(\frac{1}{q+1} \right)^{\frac{1}{q}} \left(\frac{|X'(c, \cdot)|^q + |X'(d, \cdot)|^q}{2} \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} B_6 &= \int_0^1 \frac{1}{[\theta c^p + (1 - \theta)d^p]^{r(1-\frac{1}{p})}} d\theta \\ &= \begin{cases} \frac{1}{2c^{rp-r}} {}_2F_1\left(r - \frac{r}{p}, 1; 2; 1 - \left(\frac{d}{c}\right)^p\right), & p < 0, \\ \frac{1}{2d^{rp-r}} {}_2F_1\left(r - \frac{r}{p}, 1; 2; 1 - \left(\frac{c}{d}\right)^p\right), & p > 0, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \theta |1 - 2\theta|^q d\theta &= \int_0^1 (1 - \theta) |1 - 2\theta|^q d\theta \\ &= \frac{1}{2(q+1)}. \end{aligned}$$

Remark 3.14. If we take $\alpha = 0$ in Theorem 3.13, we attain Theorem 7 in [9].

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