SAÜ Fen Bil Der 20. Cilt, 2. Sayı, s. 325-335, 2016

A study on the characterizations of non-null curves according to the Bishop frame of type-2 in Minkowski 3-space

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17.02.2016 Geliş/Received, 12.05.2016 Kabul/Accepted

ABSTRACT

In this work, we study classical differential geometry of non-null curves according to the new version of Bishop frame in E_1^3 which we call it along the work as "the Bishop frame of type-2". First, we investigate position vector of a regular and non-null curve by obtaining a system of ordinary differential equations. The solution of the system gives the components of the position vector with respect to the Bishop frame of type-2 in E_1^3. Moreover, we define the first, second and third order Bishop planes according to this new frame, and also, regardig to these planes, we characterize position vectors in E_1^3.

Keywords: spacelike curve, timelike curve, position vector, the bishop planes, the bishop frame of type-2.

Minkowski 3-uzayda 2. tip Bishop çatısına göre null olmayan eğrilerin karakterizasyonlarına dair bir inceleme

ÖZ

Bu çalışmada, E_1^3 de Bishop çatısının yeni bir yorumuna göre null olmayan eğrilerin klasik diferensiyel geometrisini inceliyoruz. Bu yeni yorumlanan çatıyı, 2. Tip Bishop çatısı şeklinde adlandırıyoruz. Öncelikle, bir adi diferensiyel denklem sistemi elde etmek suretiyle, regüler ve null olmayan eğrilerin konum vektörünü araştırıyoruz. Bu sistemin çözümü, E_1^3 de 2. tip Bishop çatısın göre konum vektörünün bileşenlerini verir. Bununla birlikte, bu yeni çatıya göre birinci, ikinci ve üçüncü mertebeden Bishop düzlemlerini tanımlıyoruz ve bu düzlemlere bağlı olarak E_1^3 de konum vektörlerini karakterize ediyoruz.

Anahtar Kelimeler: spacelike eğri, timelike eğri, konum vektörü, bishop düzlemleri, 2. tip Bishop çatısı.

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1. INTRODUCTION

The construction of the Bishop frame is due to L. R. Bishop in [2]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, an alternative frame is needed for non continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces. In applied sciences, Bishop frame is used in engineering. This special frame has been particulary used in the study of DNA, and tubular surfaces and made in robot.

By new version of Bishop frame, we mean that the tangent vector Ω_1 and principal normal vector Ω_2 are considered as parallel transport plane while the binormal vector B remains fixed. First, this new version of Bishop frame was studied in Euclidean space by Yilmaz in [15]. Then Özyilmaz gave some characterizations of curves according to this new frame in Euclidean space [10]. Also, Ünlütürk and Yılmaz obtained the new version of Bishop frame for spacelike curves in [14]. There is also a literature containing studies of curves according to Bishop frame and its new versions (see [14-17]).

A curve is thought as a geometric set of points, or locus. That is, intuitively, it can be considered as a path traced out by a particle moving in E^3. Position vectors of curves have been studied in Euclidean and its ambient spaces such as Minkowski and Galilean spaces by [1, 3, 4, 5, 10-13]. Vectorial differential equation of third order characterizes regular curves of E_1^3. Recently a method has been developed by B.Y. Chen to classify curves with solutions of differential equations with constant coefficients with respect to standard frame of the space. This method generally uses ordinary vectorial differential equations as well as Frenet equations [3]. By this way, curves of a finite Chen type and some of classifications are given by the researches in Euclidean space or another spaces, see [3].

Position vector of some special curves according to Bishop frame have been studied in E_1^3 by Yılmaz in [17]. Ali studied position vector of a timelike slant helix in E_1^3 in [1]. Additionally, Yılmaz considered position vector of partially null curve which is derived from a vectorial equation [18]. As similar to Minkowski 3-space, Yılmaz also studied position vectors of curves in Galilean 3-space G_3 [19]. Divjak considered position vectors of curves in pseudo-Galilean space G_3^1 [6].

In this work, we construct the new version of Bishop frame for a timelike curve. Along with the work, we call it as the Bishop frame of type-2. Based on this new frame, we define the Bishop planes in E_1^3. We also study classical differential geometry of timelike curves according to the Bishop frame of type-2 in E_1^3. We investigate position vector of a regular timelike curve by obtaining a system of ordinary differential equations. The solution of the system gives the components of the position vector with respect to the Bishop frame of type-2 in E_1^3.

2. PRELIMINARIES

The fundamentals of Minkowski 3-space below are cited from [8, 9].

The three dimensional Minkowski space E_1^3 is a real vector space E^3 endowed with the standard flat Lorentzian metric given by $\langle , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2$ where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. This metric is an indefinite one.

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary vectors in E_1^3, the Lorentzian cross product of u and v is defined as

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Recall that a vector $v \in E_1^3$ can have one of the following three Lorentzian characters: it is a spacelike vector if $\langle v, v \rangle > 0$ or $v = 0$; timelike $\langle v, v \rangle < 0$ and null (lightlike) $\langle v, v \rangle = 0$ for $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if its velocity vector α' are ,respectively, spacelike, timelike or null (lightlike), for every $s \in I \subset E$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|\langle a, a \rangle|}$. The curve $\alpha = \alpha(s)$ is called a unit speed curve if its velocity vector α' is unit one i.e., $\|\alpha'\| = 1$. For vectors $v, w \in E_1^3$, they are said to be orthogonal eachother if and only if $\langle v,w\rangle = 0.$ Denote by $\{T,N,B\}$ the moving Serret-Frenet frame along the curve $\alpha=\alpha(s)$ in the space E_1^3.

The Lorentzian sphere S_1^2 of radius $r>0$ and with the center in the origin of the space E_1^3 is defined by

$$S_1^2(r)=\{p=(p_1,p_2,p_3)\in E_1^3:\langle p,p\rangle = r^2\}.$$

The pseudohyperbolic space H_0^2 of radius $r>0$ and with the center in the origin of the space E_1^3 is defined by

$$H_0^2(r)=\{p=(p_1,p_2,p_3)\in E_1^3:\langle p,p\rangle = -r^2\}.$$

(i) For an arbitrary spacelike curve $\alpha=\alpha(s)$ in E_1^3, the Serret-Frenet formulae are given as follows

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \gamma\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $\gamma=\mp 1,$ and the functions κ and τ are respectively the first and second (torsion) curvature.

$$T(s)=\alpha'(s), N(s)=\frac{T'(s)}{\kappa(s)}, B(s)=T(s)\times N(s)$$

and $\tau(s)=\dfrac{\det(\alpha',\alpha'',\alpha''')}{\kappa^2(s)}.$

If $\gamma=-1,$ then $\alpha(s)$ is a spacelike curve with spacelike principal normal N and timelike binormal $B,$ its Serret-Frenet invariants are given as

$$\kappa(s)=\sqrt{\langle T'(s),T'(s)\rangle} \text{ and } \tau(s)=-\langle N'(s),B(s)\rangle.$$

If $\gamma=1,$ then $\alpha(s)$ is a spacelike curve with timelike principal normal N and spacelike binormal $B,$ also we obtain its Serret-Frenet invariants as

$$\kappa(s)=\sqrt{-\langle T'(s),T'(s)\rangle} \text{ and } \tau(s)=\langle N'(s),B(s)\rangle.$$

Theorem 2.1. ([14]), Let $\alpha=\alpha(s)$ be spacelike unit speed curve with a spacelike principal normal. If $\{\Omega_1,\Omega_2,B\}$ is adapted frame, then we have the Bishop derivative formulae as

$$\begin{bmatrix} \Omega_1' \\ \Omega_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & -\xi_2 \\ -\xi_1 & -\xi_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{2}$$

Theorem 2.2. ([14]), Let $\{T,N,B\}$ and $\{\Omega_1,\Omega_2,B\}$ be Frenet and Bishop frames, respectively. There exists a relation between them as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & \cosh\theta(s) & 0 \\ \cosh\theta(s) & \sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{3}$$

where θ is the angle between the vectors N and Ω_1. Taking the norm of both sides, we find

$$\tau=\sqrt{\left|\xi_2^2-\xi_1^2\right|}$$

and

$$\sqrt{\left|\left(\frac{\xi_1}{\tau}\right)^2-\left(\frac{\xi_2}{\tau}\right)^2\right|}=1. \tag{4}$$

By (2.4), we express

$$\xi_1=\tau(s)\cosh\theta(s), \xi_2=\tau(s)\sinh\theta(s). \tag{5}$$

The frame $\{\Omega_1,\Omega_2,B\}$ is properly oriented, and τ and $\theta(s)=\int_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha=\alpha(s)$. We shall call the set

$$\{\Omega_1,\Omega_2,B,\xi_1,\xi_2\}$$

as type-2 Bishop invariants of the curve $\alpha=\alpha(s)$ in E_1^3.

(ii) For an arbitrary timelike curve $\alpha=\alpha(s)$ in E_1^3, the Serret-Frenet formulae are given as follows

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} . \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where

$$\langle T,T\rangle=-1, \quad \langle B,B\rangle=\langle N,N\rangle=1,$$

$$T(s)=\alpha'(s), B(s)=T(s)\times N(s),$$

$$N(s)=\frac{T'(s)}{\kappa(s)}.$$

and the first and second curvatures $\kappa(s)$ and $\tau(s)$ are, respectively, given as

$$\kappa(s)=\left\|\alpha''\right\|, \quad \tau(s)=\frac{\det(\alpha',\alpha'',\alpha''')}{\kappa^2(s)}.$$

3. THE BISHOP FRAME OF TYPE-2 FOR A TIMELIKE CURVE IN E_1^3

We gave the Bishop frame of type-2 of spacelike curves in E_1^3 in [14]. So in this section, we construct the Bishop frame of type-2 of a timelike curve in E_1^3.

Theorem 3.1. Let $\alpha=\alpha(s)$ be a timelike unit speed curve with a spacelike principal normal. If $\{\Omega_1,\Omega_2,B\}$ is an adapted frame, then we have

$$\begin{bmatrix} \Omega_1' \\ \Omega_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{7}$$

Proof. Let us investigate "the Bishop frame of type-2 in E_1^3" as similar to Serret-Frenet frame, where

$$\langle\Omega_1,\Omega_1\rangle=-1, \langle B,B\rangle=\langle\Omega_2,\Omega_2\rangle=1,$$

and

$$\langle\Omega_1,\Omega_2\rangle=\langle\Omega_1,B\rangle=\langle\Omega_2,B\rangle=0.$$

If Ω_1 is a timelike vector, Ω_2 and B are spacelike vectors, then we have the equation (7) or shortly $X'=AX$. Moreover, A is a semi-skew matrix where ξ_1,ξ_2 are, respectively, first and second curvatures of the curve, and also these curvatures are defined as

$$\xi_1=\langle\Omega_1',B\rangle, \xi_2=\langle\Omega_2',B\rangle. \tag{8}$$

Theorem 3.2. Let $\{T,N,B\}$ and $\{\Omega_1,\Omega_2,B\}$ be Frenet and Bishop frames, respectively. There exists a relation between them as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sinh\theta(s) & -\cosh\theta(s) & 0 \\ \cosh\theta(s) & -\sinh\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ B \end{bmatrix} \tag{9}$$

where θ is the angle between the vectors N and Ω_1.

Proof. The tangent vector of the curve α according to the frame $\{\Omega_1,\Omega_2,B\}$ is written as

$$T=\sinh\theta(s)\Omega_1-\cosh\theta(s)\Omega_2 \tag{10}$$

and differentiating it with respect to s gives

$$\begin{aligned} T' = \kappa N = \theta'(s)[\cosh\theta(s)\Omega_1 \\ -\sinh\theta(s)\Omega_2]+\sinh\theta(s)\Omega_1' \\ -\cosh\theta(s)\Omega_2'. \end{aligned} \tag{11}$$

Substituting $\Omega_1'=\xi_1 B$ and $\Omega_2'=\xi_2 B$ into (11), we have

$$\begin{aligned} \kappa N = \theta'(s)[\cosh\theta(s)\Omega_1 \\ -\sinh\theta(s)\Omega_2]+[\sinh\theta(s)\xi_1 \\ -\cosh\theta(s)\xi_2]B. \end{aligned}$$

Also, from (8) we get

$$\theta(s)=arc\tanh\frac{\xi_2}{\xi_1}, \theta'(s)=\kappa(s), \tag{12}$$

$$N=\cosh\theta(s)\Omega_1-\sinh\theta(s)\Omega_2.$$

Since there is a solution for θ satisfying any initial condition, relatively local parallel normal fields exist. Using (7), we have

$$B'=\tau N=\xi_1\Omega_1+\xi_2\Omega_2,$$

and taking the norm of both sides, we find

$$\tau = \sqrt{\left|\xi_2^2 - \xi_1^2\right|}, \tag{13}$$

and

$$\sqrt{\left|(\frac{\xi_1}{\tau})^2 - (\frac{\xi_2}{\tau})^2\right|} = 1.$$

By (12), we express

$$\xi_1 = \tau(s)\cosh\theta(s), \xi_2 = \tau(s)\sinh\theta(s). \tag{14}$$

The frame $\{\Omega_1, \Omega_2, B\}$ is properly oriented, and τ and $\theta(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{\Omega_1, \Omega_2, B, \xi_1, \xi_2\}$ as the Bishop invariants of type-2 for the curve $\alpha = \alpha(s)$ in E_1^3.

4.APPLICATIONS OF SPACELIKE CURVES ACCORDING TO THE BISHOP FRAME OF TYPE-2

Let $\alpha = \alpha(s)$ be a spacelike curve with a spacelike principal normal. The position vector of this curve with respect to the Bishop frame of type-2 in E_1^3 as

$$\alpha = \alpha(s) = \lambda\Omega_1 + \delta\Omega_2 + \beta B, \tag{15}$$

where λ, α and β are arbitrary functions of s. Differentiating (15) and considering (2), we have a system of differential equation as follows:

$$\begin{aligned} &\lambda' - \xi_1\beta - 1 = 0, \\ &\delta' - \xi_2\beta = 0, \\ &\lambda\xi_1 - \delta\xi_2 + \beta' = 0. \end{aligned} \tag{16}$$

The system (16) characterizes the position vector of a spacelike curve according to the Bishop frame of type-2 in E_1^3. Its position vector is determined by solutions of the system (16).

Let us study the following cases for the system (16):

Case I: If $\beta = 0$, then $\alpha = \alpha(s)$ lies fully on the subspace $\Omega_1\Omega_2$. Thus we have other components as

$$\lambda = s + c_1, \quad \delta = c_2, \tag{17}$$

where c_1, c_2 are constants.

Taking (17) into consideration at $(16)_3$ gives the following linear relation among Bishop curvatures

$$\frac{\xi_1}{\xi_2} = \frac{c_2}{(s + c_1)\xi_1}. \tag{18}$$

As an immediate result, we can give the following theorem:

Theorem 4.1. Let $\alpha = \alpha(s)$ be a spacelike curve with a spacelike principal normal and lie fully in the subspace $\Omega_1\Omega_2$, then

(i) If $\beta = const.$, then α is a plane curve.

(ii) The ratio of Bishop curvatures $\frac{\xi_1}{\xi_2} = \frac{c_2}{(s + c_1)\xi_1}$ since the equation (18).

(iii) The position vector of δ is written as

$$\alpha = \alpha(s) = (s + c_1)\Omega_1 + c_2\Omega_2. \tag{19}$$

Case II: If $\lambda = const. \neq 0$, then we obtain

$$\delta = -\int_0^s \frac{\xi_2}{\xi_1} ds \text{ and } \beta = -\frac{1}{\xi_1},$$

by $(16)_1$ and $(16)_2$.

Subcase II-a: Let us suppose $\lambda = 0$, then we find again

$$\delta = -\int_0^s \frac{\xi_2}{\xi_1} ds \text{ and } \beta = -\frac{1}{\xi_1},$$

Suffice to say that this case is congruent to case II. This case yields a curve equation as follows:

$$\alpha = \alpha(s) = (-\int_0^s \frac{\xi_2}{\xi_1} ds)\Omega_2 + (-\frac{1}{\xi_1})B. \tag{20}$$

Case III: If $\delta = const. \neq 0$, then we find $\beta = 0$ and $\lambda = s + c_1$, where $c_1 \in R$, from $(16)_1$ and $(16)_2$. So this case is also congruent to case I.

Subcase III-a: If $\delta = 0$, then we straightforwardly find $\lambda = s$ and also ξ_1 and β are constant. This result follows a curve equation as

$$\alpha = \alpha(s) = (s + c_1)\Omega_1, \tag{21}$$

where $c_1 \in R$.

Theorem 4.2. The first vector field of type-2 Bishop trihedra Ω_1 satisfies a vector differential equation of third order as follows:

$$\begin{aligned}&\frac{1}{\xi_1\xi_2}\Omega_1''' + [(\frac{1}{\xi_1\xi_2})' - (\frac{\xi_1'}{\xi_1^2\xi_2})]\Omega_1'' \\ &+[\frac{\xi_2}{\xi_1} - \frac{\xi_1}{\xi_2} - (\frac{\xi_1'}{\xi_1^2\xi_2})]\Omega_1' + (\frac{\xi_1}{\xi_2})\Omega_1 = 0.\end{aligned} \tag{22}$$

Proof. Let $\alpha = \alpha(s)$ be a regular curve in E_1^3 with non-vanishing Serret-Frenet curvatures, then the equations (2) hold. By the first equation, we write

$$\Omega_1' = \xi_1 B. \tag{23}$$

Differentiating (23) gives

$$\begin{aligned}\Omega_2'' &= \frac{1}{\xi_1\xi_2}\Omega_1''' + (\frac{1}{\xi_1\xi_2})'\Omega_1'' - \\ &(\frac{\xi_1'}{\xi_1^2\xi_2})\Omega_1'' - \frac{\xi_1}{\xi_2}\Omega_1' - (\frac{\xi_1'}{\xi_1^2\xi_2})'\Omega_1' - (\frac{\xi_1}{\xi_2})'\Omega_1.\end{aligned} \tag{24}$$

From $(2)_1$ and $(2)_2$, we have

$$\Omega_2' = -\frac{\xi_2}{\xi_1}\Omega_1'. \tag{25}$$

Substituting (25) into (24) gives the equation (22).

Let α be a regular spacelike curve with non-vanishing Frenet-Serret curvatures. We may write it position vector

$$\alpha = \alpha(s) = u_1\Omega_1 + u_2\Omega_2 + u_3 B, \tag{26}$$

where u_i for $1 \le i \le 3$ are arbitrary functions of s. Differentiating (26) with respect to s, we have a system of ordinary differential equations as follows:

$$\begin{cases} u_1' + \xi_1 u_3 - \sinh\theta = 0, \\ u_2' - \xi_2 u_3 - \cosh\theta = 0, \\ u_3' + \xi_1 u_1 - \xi_2 u_2 = 0. \end{cases} \tag{27}$$

From the first equation of (27), we have

$$u_3 = \frac{1}{\xi_1}(-u_1' + \sinh\theta). \tag{28}$$

Substituting (28) into the third equation of (27) gives

$$u_2 = \frac{1}{\xi_2}(\frac{1}{\xi_1}(-u_1' + \sinh\theta))' + \frac{\xi_1}{\xi_2}u_1. \tag{29}$$

Finally, using (29) in the second equation of (27), we obtain a third order non-linear differential equation with variable coefficients as

$$\begin{aligned}&(\frac{1}{\xi_2}(\frac{1}{\xi_1}(-u_1' + \sinh\theta))' + \frac{\xi_1}{\xi_2}u_1)' \\ &-\frac{\xi_2}{\xi_1}(-u_1' + \sinh\theta) - \cosh\theta = 0.\end{aligned} \tag{30}$$

This non-linear differential equation is a characterization of spacelike curve $\alpha = \alpha(s)$. Position vector of the curve $\alpha = \alpha(s)$ can be determined by means of its solution, however the general solution of the equation (30) has not yet been found. Therefore we shall focus on some special cases as follows:

Let us suppose the components in the system (27) as

$$u_1 = \text{constant} \neq 0,$$
$$u_2 = \text{constant} \neq 0,$$
$$u_3 \neq 0.$$

By the first two equations of (27), we have

$$\begin{cases} \xi_1 u_3 - \sinh\theta = 0, \\ -\xi_2 u_3 - \cosh\theta = 0. \end{cases} \tag{31}$$

Substituting the Bishop curvatures (5) into (31) gives

$$\frac{1}{\tau}(\tanh\theta - \coth\theta) = 0, (\tau \neq 0). \tag{32}$$

By multiplicating (32) with τ, we get

$$\tanh\theta - \coth\theta = 0.$$

Definition 4.3. The subspace which is spanned by the unit vectors Ω_1 and Ω_2 is called the first type Bishop plane in E_1^3.

Theorem 4.4. Let $\alpha = \alpha(s)$ be a spacelike curve, position vector of the first type-2 Bishop plane curve α is as follows:

$$\alpha = \alpha(s) = (\int_0^s \sinh\theta ds + c_1)\Omega_1 + (\int_0^s \cosh\theta ds + c_2)\Omega_2, \quad (33)$$

where c_1, c_2 are constants.

Proof. Let $\alpha = \alpha(s)$ be a spacelike curve in E_1^3. Position vector of the first type-2 Bishop plane curve α is

$$\alpha = \alpha(s) = u_1\Omega_1 + u_2\Omega_2. \quad (34)$$

Since the curve $\alpha = \alpha(s)$ lies fully in the first type-2 Bishop plane, u_3 becomes zero. Thus the system (27) turns into

$$\begin{cases} u_1' - \sinh\theta = 0, \\ u_2' - \cosh\theta = 0, \\ \xi_1 u_1 - \xi_2 u_2 = 0. \end{cases} \quad (35)$$

From (35), we find

$$\begin{cases} u_1 = \int_0^s \sinh\theta ds + c_1, \\ u_2 = \int_0^s \cosh\theta ds + c_2. \end{cases} \quad (36)$$

Using (36) in (34), we obtain the result (33).

Definition 4.5. The subspace which is spanned by the unit vectors Ω_1 and B is called the second type Bishop plane in E_1^3.

Theorem 4.6. Let $\alpha = \alpha(s)$ be a regular spacelike curve with non-vanishing Frenet-Serret curvatures in E_1^3. If the curve $\alpha = \alpha(s)$ lies fully in the second type-2 Bishop plane, then position vector of the curve α is as follows:

$$\alpha = \alpha(s) = -\frac{1}{\xi_1}(-\frac{\cosh\theta}{\xi_2})'\Omega_1 - \frac{\cosh\theta}{\xi_2}B. \quad (37)$$

Proof. The proof of this theorem is straightforwardly seen by taking $u_2 = 0$ in the system (27).

Definition 4.7. The subspace which is spanned by the unit vectors Ω_2 and B is called the third type Bishop plane in E_1^3.

Theorem 4.8. Let $\alpha = \alpha(s)$ be a regular spacelike curve with non-vanishing Frenet-Serret curvatures in E_1^3. If the curve $\alpha = \alpha(s)$ lies fully in the third type-2 Bishop plane, then position vector of the curve α is as follows:

$$\alpha = \alpha(s) = \frac{1}{\xi_2}(\frac{\sinh\theta}{\xi_1})'\Omega_2 + \frac{\sinh\theta}{\xi_1}B. \quad (38)$$

Proof. The proof of this theorem is straightforwardly seen by taking $u_1 = 0$ in the system (27).

5.APPLICATIONS OF TIMELIKE CURVES ACCORDING TO THE BISHOP FRAME OF TYPE-2

Let $\alpha = \alpha(s)$ be a regular timelike curve with a spacelike principal normal. The position vector of this curve with respect to the Bishop frame of type-2 in E_1^3 as

$$\alpha = \alpha(s) = \lambda\Omega_1 + \delta\Omega_2 + \beta B, \quad (39)$$

where λ, α and β are arbitrary functions of s. Differentiating (39) and considering (8), we have a system of differential equation as follows:

$$\begin{aligned} &\lambda' + \xi_1\beta - 1 = 0, \\ &\delta' + \xi_2\beta = 0, \quad (40) \\ &\lambda\xi_1 + \delta\xi_2 + \beta' = 0. \end{aligned}$$

The system (40) characterizes the position vector of a timelike curve according to the Bishop frame of type-2 in E_1^3. Its position vector is determined by solutions of the system (40).

Case I: If $\beta=0,$ then $\alpha=\alpha(s)$ lies fully on the subspace $\Omega_1\Omega_2$. Thus we have other components as

$$\lambda=s+c_1,\quad \delta=const.=c_2, \tag{41}$$

where $c_1, c_2\in R.$

Taking (41) into consideration at (40) gives the following linear relation among Bishop curvatures

$$(s+c_1)\xi_1+c_2\xi_2=0. \tag{42}$$

As an immediate result, we can give the following theorem:

Theorem 5.1. Let $\alpha=\alpha(s)$ be a timelike curve with a spacelike principal normal and lie fully in the subspace $\Omega_1\Omega_2$, then

(i) If $\beta=0$, then α is a plane curve.

(ii) The second Bishop curvature $\xi_2=-\dfrac{(s+c_1)\xi_1}{c_2}$ since the equation (42).

(iii) The position vector of the curve α can be written as

$$\alpha=\alpha(s)=(s+c_1)\Omega_1+c_2\Omega_2. \tag{43}$$

Case II: If $\lambda=const.\neq 0,$ then we obtain

$$\delta=-\int_0^s\frac{\xi_2}{\xi_1}ds \text{ and } \beta=\frac{1}{\xi_1},$$

by $(40)_1$ and $(40)_2$.

Subcase II-a: Let us suppose $\lambda=0,$ then we find again

$$\delta=-\int_0^s\frac{\xi_2}{\xi_1}ds \text{ and } \beta=\frac{1}{\xi_1}.$$

Suffice to say that this case is congruent to case II. This case yields a curve equation as follows:

$$\alpha=\alpha(s)=(-\int_0^s\frac{\xi_2}{\xi_1}ds)\Omega_2+(\frac{1}{\xi_1})B. \tag{44}$$

Case III: If $\delta=const.\neq 0,$ then we find $\beta=0$ and $\lambda=s+c,$ where c is a constant by using $(40)_1$ and $(40)_2$. So this case is also congruent to case I.

Subcase III-a: If $\delta=0,$ then we straightforwardly find $\lambda=s+c$ and also ξ_1 and β are constant. This result follows a curve equation as

$$\alpha=\alpha(s)=(s+c)\Omega_1. \tag{45}$$

Theorem 5.2. Let a regular timelike curve $\alpha=\alpha(s)$ lie on the sphere H_0^2 with the center c and the radius r. The position vector of α can be written as

$$\alpha(s)-c=\frac{1}{\theta'(s)}\{-\cosh\theta\Omega_1+\sinh\theta\Omega_2-[\frac{\theta''}{\theta'(\xi_1\cosh\theta-\xi_2\sinh\theta)}]B\}. \tag{46}$$

Proof Let us suppose $\alpha=\alpha(s)$ lying on the sphere H_0^2 with the center c and the radius r. Then we write

$$\langle\alpha(s)-c,\alpha(s)-c\rangle=-r^2, \tag{47}$$

for each $s\in I\subset R.$ The equation (47) has to satisfy the contact condition. So differentiating (47) gives

$$\langle T,\delta(s)-c\rangle=0 \tag{48}$$

which means that $T\perp\alpha(s)-c.$ Then we can compose the vector $\alpha(s)-c$ with respect to the basis $\{\Omega_1,\Omega_2,B\}$ as

$$\alpha(s)-c=m_1\Omega_1+m_2\Omega_2+m_3B, \tag{49}$$

where m_i for $1\leq i\leq 3$ are arbitrary functions of s. Using the tangent vector T in (9) for $\theta=\int_0^s\kappa ds$ and (49) in (48) we have

$$\left\langle \Omega_1 \sinh\theta - \Omega_2 \cosh\theta, m_1\Omega_1 + m_2\Omega_2 + m_3 B \right\rangle = 0. \quad (50)$$

From (50), we obtain

$$\frac{m_1}{m_2} = -\coth\theta. \quad (51)$$

Differentiating (48) gives

$$\left\langle \kappa N, \alpha(s) - c \right\rangle + 1 = 0. \quad (52)$$

Substituting the principal normal vector N in (9), (12) and (49) into (52), we find

$$-\theta'(s)[m_1 \cosh\theta + m_2 \sinh\theta] - 1 = 0 \quad (53)$$

so that using (51) in (53), we obtain

$$\begin{cases} m_1 = -\dfrac{\cosh\theta}{\theta'(s)}, \\ m_2 = \dfrac{\sinh\theta}{\theta'(s)}. \end{cases} \quad (54)$$

Again differentiating (52) we have

$$\left\langle \kappa' N, \alpha(s) - c \right\rangle + \left\langle \kappa N', \alpha(s) - c \right\rangle = 0, \quad (55)$$

and also the derivative of the principal normal vector in (9) is computed as

$$\begin{aligned} N' &= \Omega_1 \sinh\theta + \Omega_2 \cosh\theta \\ &+ B(\xi_1 \cosh\theta - \xi_2 \sinh\theta). \end{aligned} \quad (56)$$

Substituting (9), (49) and (56) into (55), we find

$$m_3 = \frac{\theta''}{(\theta')^2(\xi_1 \cosh\theta - \xi_2 \sinh\theta)} \quad (57)$$

which completes the proof.

Let α be a regular timelike curve with non-vanishing Frenet-Serret curvatures. We may write its position vector

$$\alpha = \alpha(s) = u_1\Omega_1 + u_2\Omega_2 + u_3 B, \quad (58)$$

where u_i for $1 \le i \le 3$ are arbitrary functions of s. Differentiating (59) with respect to s, we have a system of ordinary differential equations as follows:

$$\begin{cases} u_1' + \xi_1 u_3 - \sinh\theta = 0, \\ u_2' + \xi_2 u_3 + \cosh\theta = 0, \\ u_3' + \xi_1 u_1 + \xi_2 u_2 = 0. \end{cases} \quad (59)$$

From the first equation of (60), we have

$$u_3 = \frac{1}{\xi_1}(-u_1' + \sinh\theta). \quad (60)$$

Substituting (61) into the third equation of (60) gives

$$u_2 = -\frac{1}{\xi_2}\left(\frac{1}{\xi_1}(-u_1' + \sinh\theta)\right)' - \frac{\xi_1}{\xi_2}u_1. \quad (61)$$

Finally, using (62) in the second equation of (60), we obtain a third order non-linear differential equation with variable coefficients as

$$\begin{aligned} &\left(-\frac{1}{\xi_2}\left(\frac{1}{\xi_1}(-u_1' + \sinh\theta)\right)' - \frac{\xi_1}{\xi_2}u_1\right)' \\ &+ \frac{\xi_2}{\xi_1}(-u_1' + \sinh\theta) + \cosh\theta = 0. \end{aligned} \quad (62)$$

This non-linear differential equation is a characterization of timelike curve $\alpha = \alpha(s)$. Position vector of the curve $\alpha = \alpha(s)$ can be determined by means of its solution, however the general solution of the equation (63) has not yet been found. Therefore we shall focus on some special cases as follows:

Let us suppose the components in the system (60) as

$$u_1 = \text{constant} \neq 0,$$
$$u_2 = \text{constant} \neq 0,$$
$$u_3 \neq 0.$$

By the first two equations of (60), we have

$$\begin{cases} \xi_1 u_3 - \sinh\theta = 0, \\ \xi_2 u_3 + \cosh\theta = 0. \end{cases} \quad (63)$$

Substituting the type-2 Bishop curvatures (3.8) into (61) gives

$$\frac{1}{\tau}(\tanh\theta - \coth\theta) = 0, (\tau \neq 0). \quad (64)$$

By multiplicating (65) with τ, we get

$$\tanh\theta + \coth\theta = 0$$

which is a contradiction.

Theorem 5.3. Let $\alpha = \alpha(s)$ be a timelike curve, position vector of the first type-2 Bishop plane regular timelike curve α is as follows:

$$\alpha = \alpha(s) = (\int_0^s \sinh\theta ds + c_1)\Omega_1 + (-\int_0^s \cosh\theta ds + c_2)\Omega_2, \quad (65)$$

where c_1, c_2 are constants.

Proof. Let $\alpha = \alpha(s)$ be a regular timelike curve in E_1^3. Position vector of the first type-2 Bishop plane regular timelike curve α is

$$\alpha = \alpha(s) = u_1\Omega_1 + u_2\Omega_2. \quad (66)$$

Since the curve $\alpha = \alpha(s)$ lies fully in the first type-2 Bishop plane, u_3 becomes zero. Thus the system (60) turns into

$$\begin{cases} u_1' - \sinh\theta = 0, \\ u_2' + \cosh\theta = 0, \\ \xi_1 u_1 + \xi_2 u_2 = 0. \end{cases} \quad (67)$$

From (68), we find

$$\begin{cases} u_1 = \int_0^s \sinh\theta ds + c_1, \\ u_2 = -\int_0^s \cosh\theta ds + c_2. \end{cases} \quad (68)$$

Substituting (69) into (67), we obtain the result (66).

Theorem 5.4. Let $\alpha = \alpha(s)$ be a regular timelike curve with non-vanishing Frenet-Serret curvatures in E_1^3. If the curve $\alpha = \alpha(s)$ lies fully in the second type-2 Bishop plane, then position vector of the curve α is as follows:

$$\alpha = \alpha(s) = -\frac{1}{\xi_1}(\frac{-\cosh\theta}{\xi_2})'\Omega_1 - \frac{\cosh\theta}{\xi_2}B. \quad (69)$$

Proof. The proof of this theorem can easily be obtained by taking $u_2 = 0$ in the system (4.21).

Theorem 5.5. Let $\alpha = \alpha(s)$ be a regular spacelike curve with non-vanishing Frenet-Serret curvatures in E_1^3. If the curve $\alpha = \alpha(s)$ lies fully in the third type-2 Bishop plane, then position vector of the curve α is as follows:

$$\alpha = \alpha(s) = -\frac{1}{\xi_2}(\frac{\sinh\theta}{\xi_1})'\Omega_2 + \frac{\sinh\theta}{\xi_1}B. \quad (70)$$

Proof. The proof of this theorem can easily be obtained by taking $u_1 = 0$ in the system (60).

Theorem 5.6. The first vector field of type-2 Bishop trihedra Ω_1 satisfies a vector differential equation of third order as follows:

$$\begin{aligned} &(\frac{\xi_2}{\xi_1})'\Omega_1''' + [2(\frac{\xi_2}{\xi_1})'' - 2\xi_1^2]\Omega_1'' \\ &+[\frac{(1-\xi_1''')\xi_2' + (\xi_2^2 - \xi_1'\xi_2 - (\xi_1\xi_2)')\xi_2}{\xi_1} \\ &+((\frac{\xi_2}{\xi_1})' - 2\xi_1^2)']\Omega_1' \\ &+(\xi_1\xi_2 + 3\xi_1\xi_1')\Omega_1 = 0. \end{aligned} \quad (71)$$

Proof. Let $\alpha = \alpha(s)$ be a regular timelike curve in E_1^3 with non-vanishing Serret-Frenet curvatures, then the Bishop derivative formulae in (7) hold. By the first equation of (7), we write

$$\Omega_1' = \xi_1 B. \quad (72)$$

Differentiating (73) gives

$$\Omega_2'' = \xi_1^2\Omega_1 + \xi_1\xi_2\Omega_2 + \xi_1' B. \tag{73}$$

Using $(7)_1$ and $(7)_2$ in (74), we have

$$\Omega_2' = \frac{\xi_2}{\xi_1}\Omega_1'. \tag{74}$$

Substituting (75) into (74) gives the equation (72).

6. CONCLUSION

The equation (72) is a non-linear vectorial differential equation which characterizes the position vector of the curve $\alpha = \alpha(s)$. It is not easy to find an analytical solution of the equation (72). If this equation can be solved, then position vector of a timelike curve with spacelike principal normal can be determined according to the Bishop frame of type-2 in E_1^3. These results may open interesting doors to special areas such as mechnical design, robotics, DNA and kinematics.

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