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SOME PROPERTIES OF NEUTROSOPHIC INTEGERS

Yılmaz ÇEVEN*, Şerife Sultan TEKİN

Department of Mathematics, Süleyman Demirel University, 32260 Isparta, Turkey

Abstract: We defined some new concepts and investigated some basic properties about neutrosophic rationals in this study. We give some divisibility properties, neutrosophic prime numbers, factorization in neutrosophic integers. As a result of investigation, we see that Division Algorithm is valid for neutrosophic integers. We define the norm of a neutrosophic integer and give some properties and relations.

Keywords: Neutrosophic ring; neutrosophic rational number; neutrosophic integer

NÖTROSOFİK TAM SAYILARIN BAZI ÖZELLİKLERİ

Özet: Bu çalışmada nötrosofik rasyonel sayılar hakkında bazı yeni kavramlar tanımlandı ve temel özellikler incelendi. Bu doğrultuda bazı bölünebilirlik özellikleri, nötrosofik asal sayı kavramı, nötrosofik tamsayıların asal çarpanlarına ayrılışı incelendi. Nötrosofik tamsayılar için bölme algoritmasının geçerli olduğu görüldü. Ayrıca bir nötrosofik tam sayının normu tanımlandı ve norm ile ilgili bazı özellikler ve bağıntılar ispatlandı.

Anahtar Kelimeler: Nötrosofik halka, nötrosofik rasyonel sayı, nötrosofik tam sayı

Corresponding author:yilmazceven@sdu.edu.tr, https://orcid.org/0000-0002-2968-1546

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1. Introduction

Florentin Smarandache introduced the neutrosophy concept in 1980. This concept is the base of many subjects as neutrosophic set, logic, probability and statistics. Neutrosophic set is a generalization of the fuzzy set.

Thanks to the neutrosophic theory, the concept of neutrosophic algebraic structures has emerged. Firstly, the concept of neutrosophic algebraic structures is introduced by V. Kandasamy and F. Smarandache in [5] and they gave some definitions about neutrosophic groups and its varieties. In [6], V. Kandasamy and F. Smarandache introduced the concept of neutrosophic rings. They worked on some of the earlier studies for rings in neutrosofic rings. Also, some researchers studied neutrosophic rings, neutrosophic triplet groups, neutrosophic triplet rings in [1,2,3,7]. In [4], Conrad gave the properties of Gaussian integers as norm, divisibility, the division theorem, Euclidean Algorithm, Bezout's theorem, unique factorization, modular arithmetic, primes.

2. Preliminaries

In this part of the manuscript, we give some elemantary definitions and results for emphasis. The reader can see [6] for further details about neutrosophic rings.

Just as complex field includes an imaginary element i with $i^2=-1$, the neutrosophic rings include the indeterminate element I satisfies the property $I^2=I$.

Definition 2.1([6]) Let (R,+, .) be a ring and I be an indeterminate element which satisfies $I^2=I$. The set

$$\langle R\cup I\rangle=\{a+bI : a,\ b\in R\}$$

is called a neutrosophic ring generated by I and R under the binary operations of R.

We denote $R\cup I$ as $R[I]$ for brevity.

Example 2.2 $\mathbb{Z}[I]$ is the neutrosophic ring of integers, $\mathbb{Q}[I]$ is the neutrosophic ring of rationals, $\mathbb{R}[I]$ is the neutrosophic ring of real numbers and $\mathbb{C}[I]$ is the neutrosophic ring of complex numbers.

Definition 2.3 The neutrosophic ring $R[I]$ is called commutative if for all $x,y\in R[I]$, $xy=yx$.

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In addition, $R[I]$ is called a commutative neutrosophic ring with unity if there exists $1 \in R[I]$ such that $1.x = x.1$ for all $x \in R[I]$.

In the following we will consider only the rings $\mathbb{Z}[I]$ and $\mathbb{Q}[I]$.

3. Results

As known, $\mathbb{Z}[I] = \{a + bI : a, b \in \mathbb{Z}\}$ and $\mathbb{Q}[I] = \{a + bI : a, b \in \mathbb{Q}\}$. $\mathbb{Z}[I]$ and $\mathbb{Q}[I]$ are commutative rings with unity $1 + 0I = 1$. It is also obvious that $\mathbb{Z} \subset \mathbb{Z}[I]$ and $\mathbb{Q} \subset \mathbb{Q}[I]$.

Proposition 3.1 Let $a + bI \in \mathbb{Z}[I]$. Then $(a + bI).(x + yI) \in \mathbb{Z}$ for $x + yI \in \mathbb{Z}[I]$ if and only if $x = (a + b)k$ and $y = -bk$ ($k \in \mathbb{Z}$).

Proof. Let $(a + bI).(x + yI) \in \mathbb{Z}$ for $x + yI \in \mathbb{Z}[I]$. Then $bx + (a + b)y = 0$. Hence we see that $x = (a + b)k$ and $y = -bk$ ($k \in \mathbb{Z}$). Converse is trivial.

Proposition 3.2 The set of invertible elements of $\mathbb{Z}[I]$ is $\{\mp 1, \mp(1 - 2I)\}$.

Proof. Let $a + bI \in \mathbb{Z}[I]$. Suppose that $a + bI$ is invertible and its inverse in $\mathbb{Z}[I]$ is $x + yI$. Then we have $(a + bI).(x + yI) = 1$. So $ax = 1$ and $bx + (a + b)y = 0$. Then we have the following case:

case 1: we have $a = x = 1$ and $b + (1 + b)y = 0$. So $y = -\dfrac{b}{1 + b} \in \mathbb{Z}$. We obtain that $b = 0,\ y = 0$ or $b = -2,\ y = -2$. Therefore $1 + 0I = 1$ and $1 - 2I$ are invertible elements.

case 2: $a = x = -1$ and $-b + (b - 1)y = 0$. So $y = \dfrac{b}{b - 1} \in \mathbb{Z}$. We obtain that $b = 0,\ y = 0$ or $b = 2,\ y = 2$. Therefore $-1 + 0I = -1$ and $-1 + 2I$ are invertible elements. Also the inverses of all elements is themselves.

Proposition 3.3 The set of invertible elements of $\mathbb{Q}[I]$ is $U = \{a + bI : a, b \in \mathbb{Q}, a \neq 0, a + b \neq 0\}$ and $(a + bI)^{-1} = \dfrac{1}{a} - \dfrac{b}{a(a + b)} I$ for $a + bI \in U$.

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Proof. Let $a+bI \in \mathbb{Q}[I]$. If $(a+bI).(x+yI)=1$ then $ax=1$ and $bx+(a+b)y=0$. Hence we have $x=\frac{1}{a}$ and $y=-\frac{b}{a(a+b)}$. Hence the claim appears to be true.

Note that since $(3-3I)(0+4I)=0$, the neutrosophic ring $\mathbb{Z}[I]$ has zero divisor. So $\mathbb{Z}[I]$ is not an integral domain.

We see that some natural properties in $\mathbb{Z}$ is not carried out faithfully by the neutrosophic ring $\mathbb{Z}[I]$. We will see this facts in the following results.

Definition 3.4 Let $x=a+bI \in \mathbb{Q}[I]$. The neutrosophic rational number $a+b-bI$ is called the conjugate of x and denoted by $\overline{x}$. The norm of the number $x=a+bI$ is defined by $N(x)=x.\overline{x}=a(a+b)$.

Proposition 3.5 Let $x=a+bI$ and $y=c+dI \in \mathbb{Q}[I]$.

(i) $N(x) \in \mathbb{Q}$,

(ii) If $x \in \mathbb{Z}[I]$, then $N(x) \in \mathbb{Z}$,

(iii) $N(x)=0 \Leftrightarrow a=0$ or $a=-b$,

(iv) $a \in \mathbb{Q} \Rightarrow N(a)=a^2$,

(v) $N(bI)=0$,

(vi) $N(xy)=N(x)N(y)$,

(vii) If $x \neq 0$ and $N(x) \neq 0$, then $\frac{1}{x} \in \mathbb{Q}[I]$,

(viii) If $y \neq 0$ and $N(y) \neq 0$, $N\left(\frac{x}{y}\right)=\frac{N(x)}{N(y)}$.

Proof. (i)-(v) is clear.

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(vi) $$\begin{aligned} N(xy) &= N(ac+(ad+bc+bd)I) \\ &= ac(ac+ad+bc+bd) \\ &= a^2c^2+a^2cd+abc^2+abcd \\ &= (a^2+ab)(c^2+cd) \\ &= N(x).N(y) \end{aligned}$$

(vii) Let $x = a+bI \in \mathbb{Q}[I]$. Then we obtain $\frac{1}{x} = \frac{\overline{x}}{x.\overline{x}} = \frac{a+b}{N(x)} - \frac{b}{N(x)} I \in \mathbb{Q}[I]$.

(viii) Let $x = a+bI \in \mathbb{Q}[I]$. Since

$$N\left(\frac{1}{x}\right) = N\left(\frac{a+b}{N(x)} - \frac{bI}{N(x)}\right)$$

$$= \frac{a+b}{N(x)}\left(\frac{a+b}{N(x)} - \frac{b}{N(x)}\right)$$

$$= \frac{1}{N(x)}$$

we have $N\left(\frac{x}{y}\right) = N\left(x.\frac{1}{y}\right) = N(x).N\left(\frac{1}{y}\right) = \frac{N(x)}{N(y)}$.

Proposition 3.6 Let $u \in \mathbb{Z}[I]$. Then u is an unit element iff $N(u) = \mp 1$.

Proof. Let u is an unit element. Then there exists an element $v \in \mathbb{Z}[I]$ such that $uv = 1$. Hence we have $N(u.v) = N(1) = 1$. By Proposition 3.5 (vi), $N(u).N(v) = 1$. So we obtain $N(u) = \mp 1$. Conversely, let $u = a+bI$ and $N(u) = a(a+b) = \mp 1$. Then it is seen that u must be one of the numbers $-1, 1, -1+2I, 1-2I$. By Proposition 3.2, u is an unit element.

Definition 3.7 If x,y are neutrosophic integers, then we write $x \mid y$, or x divides y, if there is a neutrosophic integer z such that $y = xz$. Then x and z are called divisors of y.

Example 3.8 **(i)** Since $3 = (3-2I)(1+2I)$, we have $1+2I \mid 3$ and $3-2I \mid 3$.

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(ii) Does $2+7I$ divides $3+5I$? Since $\frac{3+5I}{2+7I}=\frac{(3+5I)(9-7I)}{(2+7I)(9-7I)}=\frac{27}{18}-\frac{11}{18}I\notin\mathbb{Z}[I]$, $2+7I$ does not divide $3+5I$ in $\mathbb{Z}[I]$.

Theorem 3.9 $a+bI$ in $\mathbb{Z}[I]$ is divisible by an integer m in $\mathbb{Z}[I]$ iff $m|a$ and $m|b$ in $\mathbb{Z}$.

Proof. If $m|(a+bI)$ in $\mathbb{Z}[I]$, then we have $a+bI=m(x+yI)$ for some $x,y\in\mathbb{Z}$. So, this is equivalent to $a=mx$ and $b=my$ or $m|a$ and $m|b$ in $\mathbb{Z}$.

In Theorem 3.9, if we take $b=0$, we see that divisibility between integers does not change in $\mathbb{Z}[I]$, that is, for $a,c\in\mathbb{Z}$, $c|a$ in $\mathbb{Z}[I]$ if and only if $c|a$ in $\mathbb{Z}$. But this does not mean other properties in $\mathbb{Z}$ stay the same. For example, we will see later that all primes in $\mathbb{Z}$ factor in $\mathbb{Z}[I]$.

Theorem 3.10 For $u,v\in\mathbb{Z}[I]$, if $u|v$ in $\mathbb{Z}[I]$, then $N(u)|N(v)$ in $\mathbb{Z}$.

Proof. Write $v=uw$ for $w\in\mathbb{Z}[I]$. If we take the norm of both sides, we have $N(v)=N(u)N(w)$. So we obtain $N(u)|N(v)$ in $\mathbb{Z}$.

The converse of Theorem 3.10 is usually false. Consider $2+I$ and $3+5I$. $N(2+I)=6$ and $N(3+5I)=24$. Hence $N(2+I)|N(3+5I)$. But $2+I$ does not divides $3+5I$.

Theorem 3.11 For all $u\in\mathbb{Z}[I]$, u has at least two multiplier.

Proof. For $1\in\mathbb{Z}[I]$, we have $1=(1-2I)(1-2I)=(-1+2I)(-1+2I)=(-1)(-1)=1.1$. Then for all $u=a+bI\in\mathbb{Z}[I]$, we can write

$u=u.1=(-u)(-1)=u(1-2I)(1-2I)=u(-1+2I)(-1+2I)=(-u)(-1+2I)(1-2I)$.

Let $u\in\mathbb{Z}[I]$. There are always eight trivial factors of u: $\mp1,\mp u,\mp(1-2I),\mp(1-2I)u$. We call these the trivial factors of u. Note that the norms of these factors are ∓1 and $\mp N(u)$. Any other factor of u is called non-trivial. The non-trivial factors of u are the factors with norm strictly different from ∓1 and $\mp N(u)$.

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Definition 3.12 Let u be a neutrosophic integer. u is called a composite neutrosophic integer if it has at least two non-trivial factors. If at least one of the factors of u is a trivial factor, we call u a prime neutrosophic integer.

Example 3.13 A non-trivial factorization of 3 is $(1+2I)(3-2I)$. Generally, for a prime p in $\mathbb{Z}$, in $\mathbb{Z}[I]$, a non-trivial factorization of p is $(p+(1-p)I)(1+(p-1)I)$. So although p is prime in $\mathbb{Z}$, it is not prime in $\mathbb{Z}[I]$. Since $3+5I=(1-9I)(3-4I)$, $3+5I$ is not prime in $\mathbb{Z}[I]$. Hovewer, $5-4I$ is a prime in $\mathbb{Z}[I]$.

To show $5-4I$ is prime in $\mathbb{Z}[I]$, assume that it is a composite number and let its non-trivial factorization be $5-4I=u.v$. Then we have $N(5-4I)=N(u.v)$. So we get $5=N(u).N(v)$. Therefore $N(u)=\mp 5$ or ∓ 1. We get a contradiction since the factorization is non-trivial. Hence, $5-4I$ has only trivial factorizations in $\mathbb{Z}[I]$, so $5-4I$ is prime in $\mathbb{Z}[I]$.

So we can write the following Theorem:

Theorem 3.14. Let p be a prime number in $\mathbb{Z}$**.** If the norm of a neutrosophic integer x is $\mp p$ in $\mathbb{Z}[I]$, then x is prime in $\mathbb{Z}[I]$.

Proof. Let $u\in\mathbb{Z}[I]$ and $N(u)=p$. Assume u is a composite number and let its non-trivial factorization be $u=xy$ for $x,y\in\mathbb{Z}[I]$. Taking the norm of both sides, we have $p=N(x).N(y)$. Hence $N(x)=\mp 1$ or $\mp p$. Therefore we see that u has the trivial factorization. we get a contradiction. So a neutrosophic integer such that its norm is $\mp p$ is a prime in $\mathbb{Z}[I]$.

Corollary 3.15 The prime numbers in $\mathbb{Z}[I]$ have one of the following form:

$$\mp\left(p+(1-p)I\right),\ \mp\left(p-(1+p)I\right),\ \mp\left(1+(p-1)I\right),\ \mp\left(1-(p+1)I\right)$$

where p is a prime number in $\mathbb{Z}$.

Proof. If $N(a+bI)=a(a+b)=\mp p$, then we obtain eight different case: $a=\mp p$ and $a+b=\mp 1$, $a=\mp 1$ and $a+b=\mp p$. The desired result is obtained from these cases.

Theorem 3.16 The neutrosophic integers such that norm is an even number is a multiple of $2-I$ or $1+I$.

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Proof. Let $a+bI \in \mathbb{Z}[I]$ and $N(a+bI)=a(a+b)$ be an even number. Then, if $a=2k$ for any k in $\mathbb{Z}$, since $2k+bI=(k+(k+b)I)(2-I)$ and $N(2-I)=2$, $a+bI$ is a multiple of $2-I$. Also if $a=2k+1$ and $b=2t+1$ for any k,t in $\mathbb{Z}$, since $2k+1+(2t+1)I=(2k+1+(t-k)I)(1+I)$, and $N(1+I)=2$, $a+bI$ is a multiple of $1+I$.

Theorem 3.17 (Division Algorithm) Suppose $u,v \in \mathbb{Z}[I]$ and $v \neq 0$. Then there exist $q,r \in \mathbb{Z}[I]$ such that

$$u=q.v+r, \ \left|N(r)\right| < \left|N(v)\right|.$$

Proof. We know that $\frac{u}{v} \in \mathbb{Q}[I]$. Let $\frac{u}{v}=x+yI$ where $x,y \in \mathbb{Q}$. Let $m,n \in \mathbb{Z}$ be the nearest integers to x, y respectively. Then it can be written $\left|x-m\right| \leq \frac{1}{2}$ and $\left|y-n\right| \leq \frac{1}{2}$. Take $q=m+nI$. Then we get

$$\begin{aligned} N\left(\frac{u}{v}-q\right) &= N\big((x-m)+(y-n)I\big) \\ &= (x-m)\big((x-m)+(y-n)\big) \\ &= (x-m)^2+(x-m)(y-n) \\ &\leq \frac{1}{4}+\frac{1}{4} \\ &< 1. \end{aligned}$$

Hence since

$$\begin{aligned} \left|N\left(\frac{u}{v}-q\right)\right| &= \left|N\left(\frac{u-qv}{v}\right)\right| \\ &= \left|\frac{N(u-qv)}{N(v)}\right| \end{aligned}$$

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$$= \frac{|N(u-qv)|}{|N(v)|}$$

$$< 1,$$

we have $|N(u-qv)| < |N(v)|$. Setting $u - qv = r$, the result follows.

Example 3.18 Let $u = 5+6I, v = 3+2I.$ Then

$$\frac{u}{v} = \frac{5+6I}{3+2I} = \frac{(5+6I)(5-2I)}{(3+2I)(5-2I)} = \frac{25+8I}{15} = \frac{25}{15} + \frac{8}{15}I = x + yI.$$

Then we take $m = 2,\ n = 1$. Hence $q = 2 + I$ and $r = u - qv = -1 - 3I$. Then it is true that $u = qv + r$ and $|N(r)| = 4 < |N(v)| = 15$.

Note that there is one important difference between the Division Algorithms in $\mathbb{Z}[I]$ and $\mathbb{Z}$: the quotient and remainder are not unique in $\mathbb{Z}[I]$.

Example 3.19 We see that

$$8 + 6I = (3 + 2I)(2 + I) + 2 - 3I \text{ and } |N(2-3I)| = 2 < |N(2+I)| = 6 .$$

It is also true that

$$8 + 6I = (3 + 2I)(4 - I) - 4 + 3I \text{ and } |N(-4+3I)| = 4 < |N(4-I)| = 12 .$$

4. Conclusions

In this paper, we give some properties of neutrosophic rationals and integers. We obtain some divisibility properties, neutrosophic prime numbers, factorization of neutrosophic integers. We see that Division Algorithm is valid for neutrosophic integers. Also we see that the set of neutrosophic integers cannot be an integral domain even if the set of integers is an integral domain. In the next future, we plan some additional properties of neutrosophic integers as the Euclidean algorithm, unique factorization, modular arithmetic in $\mathbb{Z}[I]$.

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