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Research Article

Some Integral Inequalities for Harmonically (α, s)-Convex Functions

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In the paper, the author introduces a new class of harmonically convex functions, which is called harmonically (α, s)-convex functions and establishes some new integral inequalities of the Hermite-Hadamard type for harmonically (α, s)-convex functions. The properties of the newly introduced class of harmonically convex functions are also investigated. Finally, some applications to special means are given.

1. Introduction

It is well known in the literature that a function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on interval I if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1) holds in the reverse direction, f is said to be concave function.

Inequalities play a fundamental part in many branches of pure and applied sciences. A number of studies have shown that the theory of convex functions has a closely relationship with the theory of inequalities. One of the most famous inequalities for convex functions is named Hermite-Hadamard integral inequality as follows.

Suppose $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ to be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$; then the double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{2}$$

holds. If f is concave function, both inequalities in (2) are reversed. This inequality provides necessary and sufficient condition for a function to be convex.

In recent years, the concept of convexity has been improved, generalized, and extended in many directions; see [1–9]. Several new classes of convex functions have been introduced and new versions of Hermite-Hadamard's inequality have been obtained. In [10], İşcan introduced the class of harmonically convex functions and investigated the Hermite-Hadamard type inequalities for this new class of functions. For several recent results, generalizations, improvements, and refinements concerning harmonically convex functions; see [10–16]. There have been many studies dedicated to generalizing the harmonic convex functions and to establishing their Hermite-Hadamard type inequalities. For some recent studies on Hermite-Hadamard type inequalities, please refer to the monographs [10–12, 16–24].

The aim of this paper is to introduce the concept of harmonically (α, s)-convex functions and to establish several new Hermite-Hadamard type inequalities based on these new class of functions.

2. Preliminaries

In this section, we recall some basic concepts and results of harmonically convex functions.

Definition 1 ([10]). Let $I \subset (0, \infty)$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex function if

$$f\left(\frac{xy}{tx + (1-t)y}\right) = tf(y) + (1-t)f(x) \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 2 ([13]). A function $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ is said to be harmonically s-convex function in second sense, where $s \in (0,1]$, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) = t^s f(y) + (1-t)^s f(x) \quad (4)$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 3 ([15]). A function $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ is said to be harmonically s-convex function in first sense, where $s \in [0,1]$, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) = t^s f(y) + (1-t^s) f(x) \quad (5)$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 4 ([12]). The function $f : (0,b^*] \longrightarrow \mathbb{R}$, $b^* > 0$, is said to be harmonically (α,m)-convex function, where $\alpha \in [0,1]$ and $m \in (0,1]$, if

$$\begin{aligned} f\left(\frac{mxy}{mtx+(1-t)y}\right) &= f\left(\left(\frac{t}{y}+\frac{1-t}{mx}\right)^{-1}\right) \\ &\le t^\alpha f(y) + m(1-t^\alpha) f(x) \end{aligned} \quad (6)$$

for all $x, y \in (0,b^*]$ and $t \in [0,1]$.

Note that if $m = 1$ in Definition 4, we have definition of harmonically s-convex function in first sense for $\alpha = s$.

Definition 5 ([25]). A function $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ is said to be harmonically P-function if

$$f\left(\frac{xy}{tx+(1-t)y}\right) = f(x) + f(y) \quad (7)$$

for all $x, y \in I$ and $t \in [0,1]$.

Theorem 6 ([10]). *Let $I \subset (0,\infty) \longrightarrow \mathbb{R}$ be harmonically convex and $a,b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}. \quad (8)$$

Theorem 7 ([10]). *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q \ge 1$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\le \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left(\lambda_2\left|f'(a)\right|^q + \lambda_3\left|f'(b)\right|^q\right)^{1/q}, \end{aligned} \quad (9)$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right), \quad (10)$$

$$\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right), \quad (11)$$

and

$$\lambda_3 = \frac{1}{a(b-a)} + \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right). \quad (12)$$

Theorem 8 ([10]). *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $p,q > 1$, $1/p + 1/q = 1$, then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\le \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p} \\ &\cdot \left(\mu_1\left|f'(a)\right|^q + \mu_2\left|f'(b)\right|^q\right)^{1/q}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} \mu_1 &= \frac{a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q) - a\right]}{2(b-a)^2(1-q)(1-2q)}, \\ \mu_2 &= \frac{b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q) + b\right]}{2(b-a)^2(1-q)(1-2q)}. \end{aligned} \quad (14)$$

Definition 9 ([9]). Let f and g be two functions. If

$$(f(x) - f(y))(g(x) - g(y)) \ge 0 \quad (15)$$

holds for all $x, y \in \mathbb{R}$, then f and g are said to be similarly ordered functions.

Definition 10 ([23]). A function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be (α,s)-convex if

$$f(tx + (1-t)y) \le t^{\alpha s} f(x) + (1-t^\alpha)^s f(y) \quad (16)$$

for all $x, y \in I$ and $t \in (0,1)$ with $s \in [-1,1]$ and $\alpha \in (0,1]$.

In order to prove some of our main results, we need the following lemma.

Lemma 11 ([10]). *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function on I° and $a,b \in I$ with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2} \\ &\cdot \int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f'\left(\frac{ab}{tb+(1-t)a}\right)dt \end{aligned} \quad (17)$$

3. Main Results

In this section we define the concept of harmonically (α,s)-convex functions and derive our main results. Throughout this section $I \subset (0,\infty)$ is the interval and I° is the interior of I.

Definition 12. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically (α, s)-convex, where $(\alpha, s) \in (0,1]^2$, if

$$f\left(\frac{xy}{ty+(1-t)x}\right) = f\left(\left(\frac{t}{x}+\frac{1-t}{y}\right)^{-1}\right) \le t^{\alpha s} f(x) + (1-t^{\alpha})^s f(y) \tag{18}$$

for all $x, y \in I$ and $t \in (0,1)$. If the inequality in (18) is reversed, then f is said to be harmonically (α, s)-concave.

Now we give some special cases of our proposed definition of harmonically (α, s)-convex functions.

(I) If $\alpha = 1$, then we have the definition of harmonically s-convex functions in second sense.

(II) If $s = 1$, then we have the definition of harmonically $(\alpha, 1)$-convex functions or in other words harmonically s-convex functions in first sense by writing s instead of α.

(III) If $\alpha = s = 1$, then we have the definition of harmonically convex functions. Thus, every harmonically convex function is also harmonically $(1,1)$-convex function.

The following proposition is obvious.

Proposition 13. *Let $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ be a function.*

(1) *If f is (α, s)-convex and nondecreasing function, then f is harmonically (α, s)-convex.*

(2) *If f is harmonically (α, s)-convex and nonincreasing function, then f is (α, s)-convex.*

Proof. For all $t \in [0,1]$ and $x, y \in I$, we have

$$t(1-t)(x-y)^2 \ge 0. \tag{19}$$

So, the following inequality holds:

$$\frac{xy}{ty+(1-t)x} \le tx + (1-t)y. \tag{20}$$

By inequality (20), the proof is completed. □

Remark 14. According to Proposition 13, every nondecreasing convex function (or $(1,1)$-convex function) is also harmonically convex function (or harmonically $(1,1)$-convex function).

Example 15. The function $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = x$ is a nondecreasing $(1,1)$-convex function. According to Remark 14, f is harmonically $(1,1)$-convex function (or harmonically (α, s)-convex function for $\alpha = s = 1$).

Proposition 16. *Let f and g be two harmonically (α, s)-convex functions. If f and g are similarly ordered functions and $t^{\alpha s} + (1-t^{\alpha})^s \le 1$, then the product fg is also harmonically (α, s)-convex function.*

Proof. Let f and g be harmonically (α, s)-convex functions. Then

$$\begin{aligned} f\left(\frac{ab}{tb+(1-t)a}\right) g\left(\frac{ab}{tb+(1-t)a}\right) &\le \left[t^{\alpha s} f(a) + (1-t^{\alpha})^s f(b)\right] \\ &\cdot \left[t^{\alpha s} g(a) + (1-t^{\alpha})^s g(b)\right] = t^{2\alpha s} f(a) g(a) \\ &+ t^{\alpha s}(1-t^{\alpha})^s \left[f(a) g(b) + f(b) g(a)\right] \\ &+ (1-t^{\alpha})^{2s} f(b) g(b) \le t^{2\alpha s} f(a) g(a) \\ &+ t^{\alpha s}(1-t^{\alpha})^s \left[f(a) g(a) + f(b) g(b)\right] \\ &+ (1-t^{\alpha})^{2s} f(b) g(b) \\ &= \left[t^{\alpha s} f(a) g(a) + (1-t^{\alpha})^s f(b) g(b)\right] \\ &\cdot \left[t^{\alpha s} + (1-t^{\alpha})^s\right] \le t^{\alpha s} f(a) g(a) + (1-t^{\alpha})^s f(b) \\ &\cdot g(b). \end{aligned} \tag{21}$$

□

Proposition 17. *Let $f : [a,b] \subset (0,\infty) \longrightarrow \mathbb{R}$ and $g : [1/b, 1/a] \longrightarrow \mathbb{R}$ defined by $g(x) = 1/x$; then f is harmonically (α, s)-convex on $[a,b]$, where $\alpha \in (0,1]$, $s \in [0,1]$, if and only if g is (α, s)-convex on $[1/b, 1/a]$.*

Proof. Since

$$\begin{aligned} (f \circ g)(tx + (1-t)y) &= f\left(\frac{1}{tx+(1-t)y}\right) \\ &= f\left(\frac{1}{t/r + (1-t)/k}\right) = f\left(\left(\frac{t}{r}+\frac{1-t}{k}\right)^{-1}\right) \\ &\le t^{\alpha s} f(r) + (1-t^{\alpha})^s f(k) \\ &= t^{\alpha s}(f \circ g)(x) + (1-t^{\alpha})^s (f \circ g)(y) \end{aligned} \tag{22}$$

for all $x, y \in [1/b, 1/a]$, $t \in [0,1]$, where $r, k \in [a,b]$, $x = 1/r$, $y = 1/k$.

This demonstrates necessary condition is provided. Now let us show the sufficient condition is also provided.

For all $x, y \in [a,b]$ and $t \in [0,1]$, we have

$$\begin{aligned} f\left(\left(\frac{t}{x}+\frac{1-t}{y}\right)^{-1}\right) &= (f \circ g)\left(t.\frac{1}{x} + (1-t).\frac{1}{y}\right) \\ &\le t^{\alpha s}(f \circ g)\left(\frac{1}{x}\right) \\ &+ (1-t^{\alpha})^s (f \circ g)\left(\frac{1}{y}\right) \\ &= t^{\alpha s} f(x) + (1-t^{\alpha})^s f(y). \end{aligned} \tag{23}$$

This completes the proof. □

Theorem 18. *Let $f : I \longrightarrow \mathbb{R}$ be harmonically (α, s)-convex function with $\alpha \in (0,1]$, $s \in [0,1]$. If $0 < a < b < \infty$ and $f \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \le \frac{f(a)+f(b)}{2}\int_0^1 \left[t^{\alpha s} + (1-t^{\alpha})^s\right] dt. \tag{24}$$

Proof. Since f is a harmonically (α, s)-convex function, we have for all $x, y \in I$

$$f\left(\frac{xy}{ty+(1-t)x}\right) \leq t^{\alpha s} f(x) + (1-t^{\alpha})^{s} f(y) \quad (25)$$

which gives

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leq t^{\alpha s} f(a) + (1-t^{\alpha})^{s} f(b),$$
$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq t^{\alpha s} f(b) + (1-t^{\alpha})^{s} f(a) \quad (26)$$

for all $t \in [0,1]$.

Adding the above inequalities, we have

$$f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right) \leq [f(a)+f(b)]\left[t^{\alpha s} + (1-t^{\alpha})^{s}\right] \quad (27)$$

Integrating the above inequality on $[0,1]$, we obtain

$$\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt + \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt \leq [f(a)+f(b)]\int_0^1 \left[t^{\alpha s} + (1-t^{\alpha})^{s}\right]dt \quad (28)$$

which implies

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}\int_0^1 \left[t^{\alpha s} + (1-t^{\alpha})^{s}\right]dt \quad (29)$$

which is the required result. □

Remark 19. If we take $\alpha = s = 1$ in Theorem 18, then inequality (24) becomes the right-hand side of inequality (2).

If we take $\alpha = 1$ in Theorem 18, we have the following result for harmonically s-convex functions in second sense.

Corollary 20. *Let $f : I \longrightarrow \mathbb{R}$ be harmonically s-convex function in second sense where $a, b \in I$ with $a < b$ and $s \in [0,1]$. If $f \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{s+1}. \quad (30)$$

If we take $s = 0$, then Theorem 18 collapses to the following result.

Corollary 21. *Let $f : I \longrightarrow \mathbb{R}$ be harmonically P-function where $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq f(a)+f(b). \quad (31)$$

Our coming result is the Hermite-Hadamard inequality for product of two harmonically (α, s)-convex functions.

Theorem 22. *Let $f, g : I \longrightarrow \mathbb{R}$ be two harmonically (α, s)-convex functions where $a, b \in I$ with $a < b$, $\alpha \in (0,1]$, $s \in [0,1]$. If $fg \in L[a,b]$, then*

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)dx \leq \eta_1 M_1 + \eta_2 M_2 + \eta_3 M_3, \quad (32)$$

where

$$\eta_1 = f(a)g(a), \quad (33)$$

$$\eta_2 = f(b)g(b), \quad (34)$$

$$\eta_3 = f(a)g(b) + f(b)g(a), \quad (35)$$

and

$$M_1 = \int_0^1 t^{2\alpha s}dt = \frac{1}{2\alpha s+1},$$
$$M_2 = \int_0^1 (1-t^{\alpha})^{2s}dt = \beta\left(2s+1, \frac{1}{\alpha}+1\right), \quad (36)$$
$$M_3 = \int_0^1 t^{\alpha s}(1-t^{\alpha})^{s}dt.$$

β is Euler Beta function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y > 0, \quad (37)$$

and ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b,c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad (38)$$
$$c > b > 0, \ |z| < 1.$$

Proof. Let $f, g : I \longrightarrow \mathbb{R}$ be harmonically (α, s)-convex functions; then we have

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)dx &= \int_0^1 f\left(\frac{xy}{ty+(1-t)x}\right) \cdot g\left(\frac{xy}{ty+(1-t)x}\right)dt \leq \int_0^1 \left[t^{\alpha s}f(a) + (1-t^{\alpha})^{s}f(b)\right]\left[t^{\alpha s}g(a) + (1-t^{\alpha})^{s}g(b)\right]dt \\ &= \int_0^1 \left(t^{2\alpha s}f(a)g(a) + (1-t^{\alpha})^{2s}f(b)g(b) + t^{\alpha s}(1-t^{\alpha})^{s}[f(a)g(b)+f(b)g(a)]\right)dt \\ &= \eta_1\int_0^1 t^{2\alpha s}dt + \eta_2\int_0^1 (1-t^{\alpha})^{2s}dt + \eta_3\int_0^1 t^{\alpha s}(1-t^{\alpha})^{s}dt = \eta_1 M_1 + \eta_2 M_2 + \eta_3 M_3.\end{aligned} \quad (39)$$

This completes the proof. □

Theorem 23. *Under the conditions of Theorem 22, if f and g are similarly ordered functions, then we have*

$$\frac{ab}{b-a}\int_a^b\left(\frac{f(x)g(x)}{x^2}\right)dx \le \eta_1 N_1+\eta_2 N_2, \quad (40)$$

where η_1 and η_2 are given by (33) and (34),

$$N_1=\int_0^1 t^{\alpha s}dt=\frac{1}{\alpha s+1},$$
$$N_2=\int_0^1 (1-t^{\alpha})^s dt=\beta\left(s+1,\frac{1}{\alpha}+1\right). \quad (41)$$

Proof. Integrating inequality (21) completes the proof. □

Theorem 24. *Let $f: I \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonically (α,s)-convex on $[a,b]$ for $q\ge 1$, with $\alpha\in(0,1]$ and $s\in[0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left[\kappa_1\left|f'(a)\right|^q+\kappa_2\left|f'(b)\right|^q\right]^{1/q}, \quad (42)$$

where λ_1 is given by (10),

$$\kappa_1=\int_0^1\frac{|1-2t|\,t^{\alpha s}}{(tb+(1-t)a)^2}dt, \quad (43)$$

and

$$\kappa_2=\int_0^1\frac{|1-2t|\,(1-t^{\alpha})^s}{(tb+(1-t)a)^2}dt. \quad (44)$$

Proof. Using Lemma 11 and power mean inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\\
&\cdot\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt\\
&\le\frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-1/q}\\
&\times\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\right.\\
&\left.\cdot\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{1/q}\\
&\le\frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-1/q}\\
&\times\left(\frac{|1-2t|}{(tb+(1-t)a)^2}\left[t^{\alpha s}\left|f'(a)\right|^q+(1-t^{\alpha})^s\right.\right.\\
&\left.\left.\cdot\left|f'(b)\right|^q\right]dt\right)^{1/q}=\frac{ab(b-a)}{2}\\
&\cdot\lambda_1^{1-1/q}\left[\kappa_1\left|f'(a)\right|^q+\kappa_2\left|f'(b)\right|^q\right]^{1/q}.\end{aligned} \quad (45)$$

This completes the proof. □

Corollary 25. *Under the conditions of Theorem 24 if $q=1$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\left(\kappa_1\left|f'(a)\right|+\kappa_2\left|f'(b)\right|\right), \quad (46)$$

where κ_1 and κ_2 are given by (43) and (44), respectively.

Remark 26. If we take $\alpha=s=1$ in Theorem 24, then inequality (42) becomes inequality (9) of Theorem 7.

If we take $\alpha=1$ in Theorem 24, then we have result for harmonically s-convex functions in second sense.

Corollary 27. *Let $f: I \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonically s-convex in second sense on $[a,b]$ for $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left[\rho_1\left|f'(a)\right|^q+\rho_2\left|f'(b)\right|^q\right]^{1/q}, \quad (47)$$

where λ_1 is given by (10),

$$\rho_1=\int_0^1\frac{|1-2t|\,t^{s}}{(tb+(1-t)a)^2}dt, \quad (48)$$

and

$$\rho_2=\int_0^1\frac{|1-2t|\,(1-t)^s}{(tb+(1-t)a)^2}dt, \quad (49)$$

respectively.

If we take $s=1$ in Theorem 24, then we have following result for harmonically s-convex functions in first sense for $\alpha=s$.

Corollary 28. *Let $f: I \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is harmonically s-convex function in first sense, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \le \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left[\rho_1\left|f'(a)\right|^q+\rho^*\left|f'(b)\right|^q\right]^{1/q}, \quad (50)$$

where λ_1 *is given by (10), and*

$$\rho_1 = \int_0^1 \frac{|1-2t|\,t^s}{(tb+(1-t)a)^2}dt, \tag{51}$$

and

$$\rho^* = \lambda_1 - \rho_1. \tag{52}$$

If we take $s = 0$ in Theorem 24, we have the following result for harmonically P -functions.

Corollary 29. *Let* $f : I \longrightarrow \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I$ *with* $a < b$ *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically* P*-function for* $q \geq 1$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\lambda_1\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{1/q}, \tag{53}$$

where λ_1 *is given by (10).*

Theorem 30. *Let* $f : I \longrightarrow \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I^\circ$ *with* $a < b$ *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically* (α, s)*-convex on* $[a,b]$ *for* $p, q > 1$, $1/p + 1/q = 1$, *with* $\alpha \in (0,1]$, $s \in [0,1]$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p} \cdot \left(\psi_1\left|f'(a)\right|^q + \psi_2\left|f'(b)\right|^q\right)^{1/q}, \tag{54}$$

where

$$\psi_1 = \int_0^1 \frac{t^{\alpha s}}{(tb+(1-t)a)^{2q}}dt, \tag{55}$$

and

$$\psi_2 = \int_0^1 \frac{(1-t^\alpha)^s}{(tb+(1-t)a)^{2q}}dt, \tag{56}$$

respectively.

Proof. From Lemma 11, Hölder's inequality, and the fact that $|f'|^q$ is harmonically (α, s) -convex function on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab(b-a)}{2}\int_0^1 \left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt \\
&\leq \frac{ab(b-a)}{2}\left(\int_0^1 |1 - 2t|^p\,dt\right)^{1/p}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{1/q} \\
&\leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}\left[t^{\alpha s}\left|f'(a)\right|^q + (1 - t^\alpha)^s\left|f'(b)\right|^q\right]dt\right)^{1/q} \\
&= \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\psi_1\left|f'(a)\right|^q + \psi_2\left|f'(b)\right|^q\right)^{1/q}
\end{aligned} \tag{57}$$

where

$$\int_0^1 |1-2t|^p\,dt = \frac{1}{p+1}. \tag{58}$$

This completes the proof. □

Remark 31. If we take $\alpha = s = 1$ in Theorem 30, then inequality (54) becomes inequality (13) of Theorem 8.

If we take $\alpha = 1$ in Theorem 30, then we have result for harmonically s-convex functions in second sense.

Corollary 32. *Let* $f : I \longrightarrow \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I^\circ$ *with* $a < b$ *and* $f' \in L[a,b]$. *If* $|f'|^q$ *is harmonically* s*-convex function in second sense on* $[a,b]$ *for* $p, q > 1$, $1/p + 1/q = 1$, *with* $s \in [0,1]$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p} \cdot \left(\varphi_1\left|f'(a)\right|^q + \varphi_2\left|f'(b)\right|^q\right)^{1/q}, \tag{59}$$

where

$$\varphi_1 = \int_0^1 \frac{t^s}{(tb+(1-t)a)^{2q}}dt = \frac{\beta(1, s+1)}{b^{2q}} \cdot {}_2F_1\left(2q, 1, s+2, 1-\frac{a}{b}\right), \tag{60}$$

and

$$\varphi_2 = \int_0^1 \frac{(1-t)^s}{(tb+(1-t)a)^{2q}}dt = \frac{\beta(s+1, 1)}{b^{2q}} \cdot {}_2F_1\left(2q, s+1, s+2, 1-\frac{a}{b}\right), \tag{61}$$

respectively.

If we take $s = 1$ in Theorem 30, then we have the following result for harmonically s-convex functions in first sense for $\alpha = s$.

Corollary 33. *Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically s-convex in first sense on $[a,b]$ for $p, q > 1$, $1/p + 1/q = 1$, with $s \in [0,1]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p} \cdot \left(\vartheta_1 \left|f'(a)\right|^q + \vartheta_2 \left|f'(\mathrm{b})\right|^q\right)^{1/q}, \quad (62)$$

where

$$\vartheta_1 = \int_0^1 \frac{t^s}{(tb+(1-t)a)^{2q}}dt = \frac{\beta(1,s+1)}{b^{2q}} \cdot {}_2F_1\left(2q, 1, s+2, 1-\frac{a}{b}\right), \quad (63)$$

and

$$\vartheta_2 = \int_0^1 \frac{1-t^s}{(tb+(1-t)a)^{2q}}dt = \frac{b^{1-2q}-a^{1-2q}}{(b-a)(1-2q)} - \vartheta_1, \quad (64)$$

respectively.

If $s = 0$ in Theorem 30, then we have result for harmonically P-functions.

Corollary 34. *Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$ and $f' \in L[a,b]$. If $|f'|^q$ is harmonically P-function on $[a,b]$ for $p, q > 1$, $1/p + 1/q = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p} \cdot \upsilon^{1/q}\left(\left|f'(a)\right|^q + \left|f'(b)\right|^q\right)^{1/q}, \quad (65)$$

where

$$\upsilon = \int_0^1 \frac{1}{(tb+(1-t)a)^{2q}}dt = \frac{b^{1-2q}-a^{1-2q}}{(b-a)(1-2q)}. \quad (66)$$

4. Some Applications to Special Means

Let us recall the following special means for arbitrary real numbers a and b $(a \neq b)$.

(1) The arithmetic mean:

$$A \coloneqq A(a,b) = \frac{a+b}{2}. \quad (67)$$

(2) The geometric mean:

$$G \coloneqq G(a,b) = \sqrt{ab}. \quad (68)$$

(3) The p-logarithmic mean:

$$L_p \coloneqq L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, \quad p \in \mathbb{R}\setminus\{-1,0\}. \quad (69)$$

Proposition 35. *Let $0 < a < b$, $q > 1$ and $s \in (0,1)$. Then, we have*

$$\left|A\left(a^{s/q+1}, b^{s/q+1}\right) - G^2 L_{s/q-1}^{s/q-1}\right| \leq \frac{ab(b-a)(s+q)}{2q}\lambda_1^{1-1/q}\left(\rho_1 a^s + \rho_2 b^s\right)^{1/q} \quad (70)$$

where λ_1, ρ_1, and ρ_2 are given by (10), (48), and (49), respectively.

Proof. The assertion follows from inequality (47) in Corollary 27, for : $(0,\infty) \longrightarrow \mathbb{R}$, $f(x) = x^{s/q+1}/(s/q+1)$. □

Proposition 36. *Let $0 < a < b$, $p > 1$, $q = p/(p-1)$, and $s \in (0,1)$. Then, we have*

$$\left|A\left(a^{s/q+1}, b^{s/q+1}\right) - G^2 L_{s/q-1}^{s/q-1}\right| \leq \frac{ab(b-a)(s+q)}{2q}\left(\frac{1}{p+1}\right)^{1/p}\left(\vartheta_1 a^s + \vartheta_2 b^s\right)^{1/q} \quad (71)$$

where ϑ_1 and ϑ_2 are given by (63) and (64).

Proof. The assertion follows from inequality (62) in Corollary 33, for : $(0,\infty) \longrightarrow \mathbb{R}$, $f(x) = x^{s/q+1}/(s/q+1)$. □

5. Conclusion

In this paper, we introduce and investigate a new class of harmonically convex functions, which is called harmonically (α, s)-convex functions. Some new results of Hermite-Hadamard type for the newly introduced class of harmonically convex functions are established. Some applications of these results to special means have also been presented. The results of this paper may stimulate further research for the researchers working in this field.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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