Annales Mathematicae Silesianae (2020),
DOI: 10.2478/amsil-2020-0021

GENERALIZED TETRANACCI HYBRID NUMBERS

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Abstract. In this paper, we introduce the generalized Tetranacci hybrid numbers and, as special cases, Tetranacci and Tetranacci-Lucas hybrid numbers. Moreover, we present Binet's formulas, generating functions, and the summation formulas for those hybrid numbers.

1. Introduction

Hybrid numbers are a new generalization of complex, hyperbolic and dual numbers and contain complex, dual and hyperbolic numbers as well as combined and mixed states of these types of three numbers. Hybrid numbers were introduced by Özdemir [13] (see also [14]). The set of hybrid numbers will be denoted by $\mathbb{K}$ and defined by

$$\mathbb{K} = \{a+b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h} : a,b,c,d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \boldsymbol{\varepsilon}^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \varepsilon+\mathbf{i}\}.$$

This set of numbers can be thought as a set of quadruplets. The real, complex, dual and hyperbolic units are defined by

$$1 \longleftrightarrow (1,0,0,0), \mathbf{i} \longleftrightarrow (0,1,0,0), \boldsymbol{\varepsilon} \longleftrightarrow (0,0,1,0), \mathbf{h} \longleftrightarrow (0,0,0,1)$$

respectively. These units are called hybrid units.

Received: 13.12.2019.Accepted: 29.08.2020.
(2020) Mathematics Subject Classification: 11B39, 11B83, 17A45, 05A15.
Key words and phrases: Tetranacci numbers, hybrid numbers, Tetranacci hybrid numbers, Tetranacci-Lucas hybrid numbers.

The two hybrid numbers are equal if all their components are equal, one by one. The sum of two hybrid numbers is defined by summing their components. Zero is the null element. With respect to the addition operation, the inverse element of $\mathbf{Z}$ is $-\mathbf{Z}$, which is defined as having all the components of $\mathbf{Z}$ changed in their signs.

Multiplication of hybrid numbers (the Hybridian product $\mathbf{ZW} = (a_1 + b_1\mathbf{i} + c_1\boldsymbol{\varepsilon} + d_1\mathbf{h})(a_2 + b_2\mathbf{i} + c_2\boldsymbol{\varepsilon}+d_2\mathbf{h}))$ can be done according to the following Table 1:

Table 1. Multiplication Table

.	$\mathbf{1}$	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{1}$	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{1}-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}+\mathbf{1}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-\boldsymbol{\varepsilon}-\mathbf{i}$	$\boldsymbol{\varepsilon}$	1

For the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$ we list some definitions as follows (see Özdemir [13]).

- The number a is called the scalar part and is denoted by $S(\mathbf{Z})$.
- The part $b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h}$ is also called the vector part and is denoted by $V(\mathbf{Z})$.
- The conjugate of $\mathbf{Z}$, denoted by $\overline{\mathbf{Z}}$, is defined by $\overline{\mathbf{Z}} = S(\mathbf{Z}) - V(\mathbf{Z}) = a - b\mathbf{i} - c\boldsymbol{\varepsilon} - d\mathbf{h}$ as in quaternions.
- The real number

$$\mathcal{C}(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2$$

  is called the characteristic number of $\mathbf{Z}$.
- The real number

$$\Delta(\mathbf{Z}) = -(b-c)^2 + c^2 + d^2$$

  is called the type number of $\mathbf{Z}$.
- We say that

$$\begin{cases} \mathbf{Z} \text{ is elliptic} & \text{if } \Delta(\mathbf{Z}) < 0; \\ \mathbf{Z} \text{ is hyperbolic} & \text{if } \Delta(\mathbf{Z}) > 0; \\ \mathbf{Z} \text{ is parabolic} & \text{if } \Delta(\mathbf{Z}) = 0. \end{cases}$$

  These are called the types of the hybrid numbers.
- The real number

$$\|\mathbf{Z}\| = \sqrt{|\mathcal{C}(\mathbf{Z})|} = \sqrt{|a^2 + (b-c)^2 - c^2 - d^2|}$$

  is called the norm of $\mathbf{Z}$.

- The inverse of $\mathbf{Z}$ is defined by

$$\mathbf{Z}^{-1} = \frac{\overline{\mathbf{Z}}}{\mathcal{C}(\mathbf{Z})}$$

where $\|\mathbf{Z}\| \neq 0$.

Briefly $\mathbb{K}$, the set of hybrid numbers, has the following properties:

- $(\mathbb{K}, +)$ is an Abelian group.
- $\mathbb{K}$ is a non-commutative ring with respect to the addition and multiplication operations.
- Multiplication operation in $\mathbb{K}$ is associative and not commutative.
- $\mathcal{C}(\mathbf{Z}_1\mathbf{Z}_2) = \mathcal{C}(\mathbf{Z}_1)\mathcal{C}(\mathbf{Z}_2)$ for $\mathbf{Z}_1, \mathbf{Z}_2 \in \mathbb{K}$.

It is not easy to remember Table 1. Also, the lack of commutativity makes it difficult to do multiplication. A matrix representation for a hybrid number is especially important in order to facilitate multiplication of hybrid numbers. By defining an isomorphism between 2×2 matrices and hybrid numbers, it can be easily multiplied the hybrid numbers and prove many of their features. On the other hand, hybrid numbers can also be defined by considering the matrix representation.

Theorem 1.1 (Özdemir [13]). *The ring of hybrid numbers $\mathbb{K}$ is isomorphic to the ring of real 2×2 matrices $\mathbb{M}_{2\times2}$ with the map $\varphi\colon \mathbb{K} \to \mathbb{M}_{2\times2}$ where*

$$\varphi(a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}) = \begin{pmatrix} a+c & b-c+d \\ c-b+d & a-c \end{pmatrix}$$

for $\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h} \in \mathbb{K}$.

We denote the matrix given in Theorem 1.1 by $A = \begin{pmatrix} a+c & b-c+d \\ c-b+d & a-c \end{pmatrix}$ for $\mathbf{Z} = a + b\mathbf{i} + c\boldsymbol{\varepsilon} + d\mathbf{h}$. The matrix $\varphi(\mathbf{Z}) = A \in \mathbb{M}_{2\times2}(\mathbb{R})$ is called the hybrid matrix corresponding to the hybrid number $\mathbf{Z}$. Note that we have

$$\varphi^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{a+d}{2}\right) + \left(\frac{a+b-c-d}{2}\right)\mathbf{i} + \left(\frac{a-d}{2}\right)\boldsymbol{\varepsilon} + \left(\frac{b+c}{2}\right)\mathbf{h}.$$

From the above isomorphism we have the matrix representations

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varphi(\mathbf{i}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \varphi(\boldsymbol{\varepsilon}) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \varphi(\mathbf{h}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With the aid of these four matrices, the multiplication of the hybrid numbers described above, can also be easily handled. It can easily done operations and calculations in the hybrid numbers using the corresponding matrices

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \boldsymbol{\varepsilon} \leftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{h} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The hybrid numbers can be classified with respect to determinant and discriminant of the characteristic equation of the 2×2 corresponding matrix.

The classification of hybrid numbers depends entirely on the determinant and the trace of the 2×2 corresponding matrix (for more details, see Özdemir [13, 14]).

Theorem 1.2 (Özdemir [13]). *Let A be a 2×2 real matrix corresponding to the hybrid number $\mathbf{Z}$. Then the followings hold.*
(a) $\|\mathbf{Z}\| = \sqrt{|\det A|}$
(b) $\Delta(\mathbf{Z}) = \left(\frac{\operatorname{tr} A}{2}\right)^2 - \det A$
(c) $\mathbf{Z}^{-1}$ *exists if and only if* $\det(A) \neq 0$.

For more details about these hybrid numbers, we refer to the works of [6, 13, 14].

A generalized Tetranacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4} \qquad (1.1)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

This sequence has been studied by many authors and more details can be found in the extensive literature dedicated to these sequences, see for example [9, 11, 12, 15, 19, 20].

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} + V_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (1.1) holds for all integers n.

The first few generalized Tetranacci numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few generalized Tetranacci numbers

n	0	1	2	3	4	5	$\ldots$
V_n	c_0	c_1	c_2	c_3	$c_0 + c_1 + c_2 + c_3$	$c_0 + 2c_1 + 2c_2 + 2c_3$	$\ldots$
V_{-n}	c_0	$c_3 - c_2 - c_1 - c_0$	$2c_2 - c_3$	$2c_1 - c_2$	$2c_0 - c_1$	$2c_3 - 2c_2 - 2c_1 - 3c_0$	$\ldots$

If we set $V_0 = 0, V_1 = 1, V_2 = 1, V_3 = 2$, then $\{V_n\}$ is the well-known Tetranacci sequence and if we set $V_0 = 3, V_1 = 1, V_2 = 3, V_3 = 7$ then $\{V_n\}$ is the well-known Tetranacci-Lucas sequence. In other words, Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$(1.2)\quad M_n = M_{n-1}+M_{n-2}+M_{n-3}+M_{n-4}, \quad M_0 = 0, M_1 = M_2 = 1, M_3 = 2$$

and

$$(1.3)\quad R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \quad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7.$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2) and (1.3) hold for all integers n. Next, we present the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts, in the following Table 3.

Table 3. A few values of Tetranacci and Tetranacci-Lucas numbers

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
M_n	1	-3	2	0	0	-1	1	0	0	0	1	1	2	4	8	15	29	56	108
R_n	-19	15	-1	-1	-6	7	-1	-1	-1	4	1	3	7	15	26	51	99	191	367

It is well known that for all integers n, usual Tetranaci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

(see for example [9] or [21]) or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1}$$

(see for example [7]) and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

respectively, where α, β, γ and δ are the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\alpha = \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\beta = \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\gamma = \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

$$\delta = \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

We present Binet's formula of the generalized Tetranacci sequence.

Corollary 1.3. *The Binet's formula of the generalized Tetranacci sequence $\{V_n\}$ is given as*

$$V_n = A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\delta^{n-6},$$

where

$$A = \frac{\alpha - 1}{5\alpha - 8}(V_3\alpha^3 + (V_0 + V_1 + V_2)\alpha^2 + (V_1 + V_2)\alpha + V_2),$$

$$B = \frac{\beta - 1}{5\beta - 8}(V_3\beta^3 + (V_0 + V_1 + V_2)\beta^2 + (V_1 + V_2)\beta + V_2),$$

$$C = \frac{\gamma - 1}{5\gamma - 8}(V_3\gamma^3 + (V_0 + V_1 + V_2)\gamma^2 + (V_1 + V_2)\gamma + V_2),$$

$$D = \frac{\delta - 1}{5\delta - 8}(V_3\delta^3 + (V_0 + V_1 + V_2)\delta^2 + (V_1 + V_2)\delta + V_2).$$

PROOF. For a proof see [16, Corollary 1.3]. Some other proofs can be found in the literature. Note that the usual Binet formula of generalized Fibonacci numbers (which includes generalized Tetranacci numbers) is largely studied in the literature (see for example [2, 3, 4, 8] and references therein). □

In fact, Corollary 1.3 is a special case of a result in [1, Remark 2.3].

Note that the Binet form of a sequence satisfying (1.1) for non-negative integers is valid for all integers n, for a proof of this result see [10]. This result of Howard and Saidak [10] is even true in the case of higher-order recurrence relations.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

LEMMA 1.4. *Suppose that* $f_{V_n}(x) = \sum_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized Tetranacci sequence* $\{V_n\}_{n\geq 0}$*. Then* $f_{V_n}(x)$ *is given by*

$$f_{V_n}(x) = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3}{1 - x - x^2 - x^3 - x^4}. \tag{1.4}$$

PROOF. Using (1.1) and some calculation, we obtain

$$\begin{aligned} &f_{V_n}(x) - x f_{V_n}(x) - x^2 f_{V_n}(x) - x^3 f_{V_n}(x) - x^4 f_{V_n}(x) \\ &\qquad = V_0 + (V_1 - V_0)x + (V_2 - V_1 - V_0)x^2 + (V_3 - V_2 - V_1 - V_0)x^3 \end{aligned}$$

which gives (1.4). □

The previous Lemma gives the following results as particular examples: generating function of the Tetranacci sequence M_n is

$$f_{M_n}(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4}$$

and generating function of the Tetranacci-Lucas sequence R_n is

$$f_{R_n}(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{4 - 3x - 2x^2 - x^3}{1 - x - x^2 - x^3 - x^4}.$$

2. Generalized Tetranacci hybrid numbers and their generating functions, Binet's formulas and summations formulas

In this section, we define generalized Tetranacci hybrid numbers and give generating functions, Binet formulas, summations formulas for them. As special cases, we present generating functions, Binet formulas, summations formulas for Tetranacci and Tetranacci-Lucas hybrid numbers.

First, we give some information about hybrid number sequences from the literature. Szynal-Liana [17] introduced nth Horadam hybrid numbers as

$$H_n = W_n + \mathbf{i}W_{n+1} + \varepsilon W_{n+2} + \mathbf{h}W_{n+3}$$

where W_n are the Horadam numbers given by the second order recurrence relation $W_n = pW_{n-1} - qW_{n-2}$ with initial values W_0, W_1 and p, q, n are integers. Szynal-Liana and Włoch [18] and Cerda-Morales [5] also studied Horadam types hybrid numbers.

We now define generalized Tetranacci hybrid numbers over the hybridian algebra $\mathbb{K}$.

DEFINITION 2.1. The nth generalized Tetranacci hybrid number is

$$\mathbb{H}V_n = V_n + V_{n+1}\mathbf{i} + V_{n+2}\varepsilon + V_{n+3}\mathbf{h}. \tag{2.1}$$

As special cases, the nth Tetranacci hybrid number and the nth Tetranacci-Lucas hybrid number are given as

$$\mathbb{H}M_n = M_n + M_{n+1}\mathbf{i} + M_{n+2}\varepsilon + M_{n+3}\mathbf{h}$$

and

$$\mathbb{H}R_n = R_n + R_{n+1}\mathbf{i} + R_{n+2}\varepsilon + R_{n+3}\mathbf{h}$$

respectively.

Note that, by definition, $\mathbb{H}V_n$ is well-defined. It can be easily shown that $\{\mathbb{H}V_n\}_{n\geq 0}$ can also be defined by the recurrence relation:

$$\mathbb{H}V_n = \mathbb{H}V_{n-1} + \mathbb{H}V_{n-2} + \mathbb{H}V_{n-3} + \mathbb{H}V_{n-4} \tag{2.2}$$

with the intial conditions $\mathbb{H}V_0$, $\mathbb{H}V_1$, $\mathbb{H}V_2$, $\mathbb{H}V_3$ (see Table 4).

Note that the sequence $\{\mathbb{H}V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathbb{H}V_{-n} = -\mathbb{H}V_{-(n-1)} - \mathbb{H}V_{-(n-2)} - \mathbb{H}V_{-(n-3)} + \mathbb{H}V_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.2) holds for all integers n.

The first few generalized Tetranacci hybrid numbers with positive subscript and negative subscript are given in the following Table 4:

Table 4. Generalized Tetranacci hybrid numbers

n	$\mathbb{H}V_n$
-5	$(2c_3 - 2c_2 - 2c_1 - 3c_0) + (2c_0 - c_1)\mathbf{i} + (2c_1 - c_2)\boldsymbol{\varepsilon} + (2c_2 - c_3)\mathbf{h}$
-4	$(2c_0 - c_1) + (2c_1 - c_2)\mathbf{i} + (2c_2 - c_3)\boldsymbol{\varepsilon} + \mathbf{h}(c_3 - c_2 - c_1 - c_0)$
-3	$(2c_1 - c_2) + (2c_2 - c_3)\mathbf{i} + (c_3 - c_2 - c_1 - c_0)\boldsymbol{\varepsilon} + c_0\mathbf{h}$
-2	$(2c_2 - c_3) + (c_3 - c_2 - c_1 - c_0)\mathbf{i} + c_0\boldsymbol{\varepsilon} + c_1\mathbf{h}$
-1	$(c_3 - c_2 - c_1 - c_0) + c_0\mathbf{i} + c_1\boldsymbol{\varepsilon} + c_2\mathbf{h}$
0	$c_0 + c_1\mathbf{i} + c_2\boldsymbol{\varepsilon} + c_3\mathbf{h}$
1	$c_1 + c_2\mathbf{i} + c_3\boldsymbol{\varepsilon} + (c_0 + c_1 + c_2 + c_3)\mathbf{h}$
2	$c_2 + c_3\mathbf{i} + (c_0 + c_1 + c_2 + c_3)\boldsymbol{\varepsilon} + (c_0 + c_1 + c_2 + c_3)\mathbf{h}$
3	$c_3 + (c_0 + c_1 + c_2 + c_3)\mathbf{i} + (c_0 + 2c_1 + 2c_2 + 2c_3)\boldsymbol{\varepsilon} + (2c_0 + 3c_1 + 4c_2 + 4c_3)\mathbf{h}$
4	$(c_0 + c_1 + c_2 + c_3) + (c_0 + 2c_1 + 2c_2 + 2c_3)\mathbf{i} + (2c_0 + 3c_1 + 4c_2 + 4c_3)\boldsymbol{\varepsilon}$ $+(4c_0 + 6c_1 + 7c_2 + 8c_3)\mathbf{h}$
5	$(c_0 + 2c_1 + 2c_2 + 2c_3) + (2c_0 + 3c_1 + 4c_2 + 4c_3)\mathbf{i} + (4c_0 + 6c_1 + 7c_2 + 8c_3)\boldsymbol{\varepsilon}$ $+(8c_0 + 12c_1 + 14c_2 + 15c_3)\mathbf{h}$

The first few Tetranacci and Tetranacci-Lucas hybrid numbers with positive subscript and negative subscript are given in Table 5 and Table 6.

Table 5. Tetranacci hybrid numbers

n	$\mathbb{H}M_n$	$\mathbb{H}M_{-n}$
0	$\mathbf{i} + \boldsymbol{\varepsilon} + 2\mathbf{h}$	$\mathbf{i} + \boldsymbol{\varepsilon} + 2\mathbf{h}$
1	$1 + \mathbf{i} + 2\boldsymbol{\varepsilon} + 4\mathbf{h}$	$\boldsymbol{\varepsilon} + \mathbf{h}$
2	$1 + 2\mathbf{i} + 4\boldsymbol{\varepsilon} + 8\mathbf{h}$	$\mathbf{h}$
3	$2 + 4\mathbf{i} + 8\boldsymbol{\varepsilon} + 15\mathbf{h}$	1
4	$4 + 8\mathbf{i} + 15\boldsymbol{\varepsilon} + 29\mathbf{h}$	$-1 + \mathbf{i}$
5	$8 + 15\mathbf{i} + 29\boldsymbol{\varepsilon} + 56\mathbf{h}$	$-\mathbf{i} + \boldsymbol{\varepsilon}$
6	$15 + 29\mathbf{i} + 56\boldsymbol{\varepsilon} + 108\mathbf{h}$	$-\boldsymbol{\varepsilon} + \mathbf{h}$
7	$29 + 56\mathbf{i} + 108\boldsymbol{\varepsilon} + 208\mathbf{h}$	$2 - \mathbf{h}$

For two generalized Tetranacci hybrid numbers $\mathbb{H}V_n$ and $\mathbb{H}V_k$, the addition and substraction are defined as componentwise, i.e.,

$$\mathbb{H}V_n + \mathbb{H}V_k = (V_n + V_k) + (V_{n+1} + V_{k+1})\mathbf{i} + (V_{n+2} + V_{k+2})\boldsymbol{\varepsilon} + (V_{n+3} + V_{k+3})\mathbf{h},$$

$$\mathbb{H}V_n - \mathbb{H}V_k = (V_n - V_k) + (V_{n+1} - V_{k+1})\mathbf{i} + (V_{n+2} - V_{k+2})\boldsymbol{\varepsilon} + (V_{n+3} - V_{k+3})\mathbf{h},$$

respectively.

Table 6. Tetranacci-Lucas hybrid numbers

n	$\mathbb{H}R_n$	$\mathbb{H}R_{-n}$
0	$4+\mathbf{i}+3\boldsymbol{\varepsilon}+7\mathbf{h}$	$4+\mathbf{i}+3\boldsymbol{\varepsilon}+7\mathbf{h}$
1	$1+3\mathbf{i}+7+15\mathbf{h}$	$-1+4\mathbf{i}+\boldsymbol{\varepsilon}+3\mathbf{h}$
2	$3+7\mathbf{i}+15\boldsymbol{\varepsilon}+26\mathbf{h}$	$-1-\mathbf{i}+4\boldsymbol{\varepsilon}+\mathbf{h}$
3	$7+15\mathbf{i}+26\boldsymbol{\varepsilon}+51\mathbf{h}$	$-1-\mathbf{i}-\boldsymbol{\varepsilon}+4\mathbf{h}$
4	$15+26\mathbf{i}+51\boldsymbol{\varepsilon}+99\mathbf{h}$	$7-\mathbf{i}-\boldsymbol{\varepsilon}-\mathbf{h}$
5	$26+51\mathbf{i}+99\boldsymbol{\varepsilon}+191\mathbf{h}$	$-6+7\mathbf{i}-\boldsymbol{\varepsilon}-\mathbf{h}$
6	$51+99\mathbf{i}+191\boldsymbol{\varepsilon}+367\mathbf{h}$	$-1-6\mathbf{i}+7\boldsymbol{\varepsilon}-\mathbf{h}$
7	$99+191\mathbf{i}+367\boldsymbol{\varepsilon}+708\mathbf{h}$	$-1-\mathbf{i}-6\boldsymbol{\varepsilon}+7\mathbf{h}$

Now, we will state Binet's formula for the generalized Tetranacci hybrid numbers and in the rest of the paper we fix the following notations:

$$\widehat{\alpha}=1+\alpha\mathbf{i}+\alpha^2\boldsymbol{\varepsilon}+\alpha^3\mathbf{h},$$

$$\widehat{\beta}=1+\beta\mathbf{i}+\beta^2\boldsymbol{\varepsilon}+\beta^3\mathbf{h},$$

$$\widehat{\gamma}=1+\gamma\mathbf{i}+\gamma^2\boldsymbol{\varepsilon}+\gamma^3\mathbf{h},$$

$$\widehat{\delta}=1+\delta\mathbf{i}+\delta^2\boldsymbol{\varepsilon}+\delta^3\mathbf{h}.$$

THEOREM 2.2 (Binet's Formula). *For any integer n, the nth generalized Tetranacci hybrid number is*

$$\mathbb{H}V_n=A\widehat{\alpha}\alpha^{n-6}+B\widehat{\beta}\beta^{n-6}+C\widehat{\gamma}\gamma^{n-6}+D\widehat{\delta}\delta^{n-6} \tag{2.3}$$

where A,B,C and D are as in Corollary 1.3.

PROOF. Using Binet's formula of the generalized Tetranacci numbers, we obtain

$$\begin{aligned}\mathbb{H}V_n &= V_n+V_{n+1}\mathbf{i}+V_{n+2}\boldsymbol{\varepsilon}+V_{n+3}\mathbf{h}\\ &= A\alpha^{n-6}+B\beta^{n-6}+C\gamma^{n-6}+D\delta^{n-6}\\ &\quad+(A\alpha^{n-5}+B\beta^{n-5}+C\gamma^{n-5}+D\delta^{n-5})\mathbf{i}\\ &\quad+(A\alpha^{n-4}+B\beta^{n-4}+C\gamma^{n-4}+D\delta^{n-4})\boldsymbol{\varepsilon}\\ &\quad+(A\alpha^{n-3}+B\beta^{n-3}+C\gamma^{n-3}+D\delta^{n-3})\mathbf{k}\\ &= A\widehat{\alpha}\alpha^{n-6}+B\widehat{\beta}\beta^{n-6}+C\widehat{\gamma}\gamma^{n-6}+D\widehat{\delta}\delta^{n-6}.\end{aligned}$$

This proves (2.3). □

As special cases, for any integer n, the Binet's Formula of nth Tetranacci hybrid number is

$$\mathbb{H}M_n = \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1}$$

and the Binet's Formula of nth Tetranacci-Lucas hybrid number is

$$\mathbb{H}R_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n.$$

Next, we present generating functions.

THEOREM 2.3. *The generating function for the generalized Tetranacci hybrid numbers is*

$$\sum_{n=0}^{\infty} \mathbb{H}V_n x^n = \frac{\mathbb{H}V_0 + (\mathbb{H}V_1 - \mathbb{H}V_0)x + (\mathbb{H}V_2 - \mathbb{H}V_1 - \mathbb{H}V_0)x^2 + \mathbb{H}V_{-1}x^3}{1-x-x^2-x^3-x^4}.$$

PROOF. Let

$$g(x) = \sum_{n=0}^{\infty} \mathbb{H}V_n x^n$$

be generating function of the generalized Tetranacci hybrid numbers. Then using the definition of the Tetranacci hybrid numbers, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$ and using the recurrence relation $\mathbb{H}V_n = \mathbb{H}V_{n-1} + \mathbb{H}V_{n-2} + \mathbb{H}V_{n-3} + \mathbb{H}V_{n-4}$, we obtain

$$(1-x-x^2-x^3-x^4)g(x) = \mathbb{H}V_0 + (\mathbb{H}V_1 - \mathbb{H}V_0)x + (\mathbb{H}V_2 - \mathbb{H}V_1 - \mathbb{H}V_0)x^2 + (\mathbb{H}V_3 - \mathbb{H}V_2 - \mathbb{H}V_1 - \mathbb{H}V_0)x^3.$$

Rearranging above equation and using $\mathbb{H}V_3 = \mathbb{H}V_2 + \mathbb{H}V_1 + \mathbb{H}V_0 + \mathbb{H}V_{-1}$, we get

$$g(x) = \frac{\mathbb{H}V_0 + (\mathbb{H}V_1 - \mathbb{H}V_0)x + (\mathbb{H}V_2 - \mathbb{H}V_1 - \mathbb{H}V_0)x^2 + \mathbb{H}V_{-1}x^3}{1-x-x^2-x^3-x^4}. \qquad \square$$

As special cases, the generating functions for the Tetranacci and Tetranacci-Lucas hybrid numbers are

$$\sum_{n=0}^{\infty} \mathbb{H}M_n x^n = \frac{(\mathbf{i}+\varepsilon+2\mathbf{h}) + (1+\varepsilon+2\mathbf{h})x + (\varepsilon+2\mathbf{h})x^2 + (\varepsilon+\mathbf{h})x^3}{1-x-x^2-x^3-x^4}$$

and

$$\sum_{n=0}^{\infty} \mathbb{H}R_n x^n = \frac{(4+\mathbf{i}+3\boldsymbol{\varepsilon}+7\mathbf{h})+(-3+2\mathbf{i}+4\boldsymbol{\varepsilon}+8\mathbf{h})x}{1-x-x^2-x^3-x^4} + \frac{(-2+3\mathbf{i}+5\boldsymbol{\varepsilon}+4\mathbf{h})x^2+(-1+4\mathbf{i}+\boldsymbol{\varepsilon}+3\mathbf{h})x^3}{1-x-x^2-x^3-x^4},$$

respectively.

Next we present some summation formulas of generalized Tetranacci numbers.

LEMMA 2.4. *For $n \geq 1$, we have the following formulas:*

(a) $\sum_{p=1}^{n} V_p = \frac{1}{3}(V_{n+2}+2V_n+V_{n-1}-V_0+V_1-V_3)$,

(b) $\sum_{p=1}^{n} V_{2p+1} = \frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0-V_1-3V_2+V_3)$,

(c) $\sum_{p=1}^{n} V_{2p} = \frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+V_0-V_1+3V_2-2V_3)$.

The above Lemma is given in Soykan [16, Theorem 2.6].

Note that from the above Lemma we have

$$\begin{aligned}\sum_{p=0}^{n} V_p &= V_0 + \sum_{p=1}^{n} V_p = V_0 + \frac{1}{3}(V_{n+2}+2V_n+V_{n-1}-V_0+V_1-V_3) \\ &= \frac{1}{3}(V_{n+2}+2V_n+V_{n-1}+2V_0+V_1-V_3),\end{aligned}$$

$$\begin{aligned}\sum_{p=0}^{n} V_{2p+1} &= V_1 + \sum_{p=1}^{n} V_{2p+1} \\ &= V_1 + \frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0-V_1-3V_2+V_3) \\ &= \frac{1}{3}(2V_{2n+2}+V_{2n}-V_{2n-1}-2V_0+2V_1-3V_2+V_3), \quad (2.4)\end{aligned}$$

and

$$\begin{aligned}\sum_{p=0}^{n} V_{2p} &= V_0 + \sum_{p=1}^{n} V_{2p} \\ &= V_0 + \frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+V_0-V_1+3V_2-2V_3) \\ &= \frac{1}{3}(2V_{2n+1}+V_{2n-1}-V_{2n-2}+4V_0-V_1+3V_2-2V_3). \quad (2.5)\end{aligned}$$

In the following Theorem, we give some summation formulas of generalized Tetranacci hybrid numbers.

THEOREM 2.5. *For $n \geq 0$, we have the following formulas:*
(a)

$$\sum_{p=0}^{n} \mathbb{H}V_p = \frac{1}{3}(\mathbb{H}V_{n+2} + 2\mathbb{H}V_n + \mathbb{H}V_{n-1} + c), \tag{2.6}$$

where

$$\begin{aligned} c &= 2V_0 + V_1 - V_3 + (-V_0 + V_1 - V_3)\mathbf{i} \\ &\quad + (-V_0 - 2V_1 - V_3)\boldsymbol{\varepsilon} + (-V_0 - 2V_1 - 3V_2 - V_3)\mathbf{h}. \end{aligned}$$

(b)

$$\sum_{p=0}^{n} \mathbb{H}V_{2p+1} = \frac{1}{3}(2\mathbb{H}V_{2n+2} + \mathbb{H}V_{2n} - \mathbb{H}V_{2n-1} + d),$$

where

$$\begin{aligned} d &= (-2V_0 + 2V_1 - 3V_2 + V_3) + (V_0 - V_1 + 3V_2 - 2V_3)\mathbf{i} \\ &\quad + (-2V_0 - V_1 - 3V_2 + V_3)\boldsymbol{\varepsilon} + (V_0 - V_1 - 2V_3)\mathbf{h}. \end{aligned}$$

(c)

$$\sum_{p=0}^{n} \mathbb{H}V_{2p} = \frac{1}{3}(2\mathbb{H}V_{2n+1} + \mathbb{H}V_{2n-1} - \mathbb{H}V_{2n-2} + e),$$

where

$$\begin{aligned} e &= (4V_0 - V_1 + 3V_2 - 2V_3) + (-2V_0 + 2V_1 - 3V_2 + V_3)\mathbf{i} \\ &\quad + (V_0 - V_1 + 3V_2 - 2V_3)\boldsymbol{\varepsilon} + (-2V_0 - V_1 - 3V_2 + V_3)\mathbf{h}. \end{aligned}$$

PROOF. (a) Using (2.1), we obtain

$$\begin{aligned} \sum_{p=0}^{n} \mathbb{H}V_p &= \sum_{p=0}^{n} V_p + \mathbf{i}\sum_{p=0}^{n} V_{p+1} + \boldsymbol{\varepsilon}\sum_{p=0}^{n} V_{p+2} + \mathbf{h}\sum_{p=0}^{n} V_{p+3} \\ &= (V_0 + ... + V_n) + \mathbf{i}(V_1 + ... + V_{n+1}) \\ &\quad + \boldsymbol{\varepsilon}(V_2 + ... + V_{n+2}) + \mathbf{h}(V_3 + ... + V_{n+3}), \end{aligned}$$

and so,

$$\begin{aligned} 3\sum_{p=0}^{n} \mathbb{H}V_p &= (V_{n+2} + 2V_n + V_{n-1} + 2V_0 + V_1 - V_3) \\ &\quad + \mathbf{i}(V_{n+3} + 2V_{n+1} + V_n + 2V_0 + V_1 - V_3 - 3V_0) \end{aligned}$$

$$+\,\boldsymbol{\varepsilon}(V_{n+4}+2V_{n+2}+V_{n+1}+2V_0+V_1-V_3-3(V_0+V_1))$$
$$+\,k(V_{n+5}+2V_{n+3}+V_{n+2}+2V_0+V_1-V_3-3(V_0+V_1+V_2))$$
$$=\mathbb{H}V_{n+2}+2\mathbb{H}V_n+\mathbb{H}V_{n-1}+c,$$

where

$$\begin{aligned} c &= 2V_0+V_1-V_3+\mathbf{i}(2V_0+V_1-V_3-3V_0) \\ &\quad +\boldsymbol{\varepsilon}(2V_0+V_1-V_3-3(V_0+V_1))+\mathbf{h}(2V_0+V_1-V_3-3(V_0+V_1+V_2)) \\ &= 2V_0+V_1-V_3+\mathbf{i}(-V_0+V_1-V_3) \\ &\quad +\boldsymbol{\varepsilon}(-V_0-2V_1-V_3)+\mathbf{h}(-V_0-2V_1-3V_2-V_3). \end{aligned}$$

Hence

$$\sum_{p=0}^{n}\mathbb{H}V_p=\frac{1}{3}(\mathbb{H}V_{n+2}+2\mathbb{H}V_n+\mathbb{H}V_{n-1}+c).$$

This proves (2.6).

(b) and (c) follows from the identities (2.4) and (2.5). □

As special cases, we have the following two corollaries.

Corollary 2.6. *For $n\geq 0$, we have the following formulas:*

(a) $\displaystyle\sum_{p=0}^{n}\mathbb{H}M_p=\frac{1}{3}(\mathbb{H}M_{n+2}+2\mathbb{H}M_n+\mathbb{H}M_{n-1}-(1+\mathbf{i}+4\boldsymbol{\varepsilon}+7\mathbf{h})),$

(b) $\displaystyle\sum_{p=0}^{n}\mathbb{H}M_{2p+1}=\frac{1}{3}(2\mathbb{H}M_{2n+2}+\mathbb{H}M_{2n}-\mathbb{H}M_{2n-1}+(1-2\mathbf{i}-2\boldsymbol{\varepsilon}-5\mathbf{h})),$

(c) $\displaystyle\sum_{p=0}^{n}\mathbb{H}M_{2p}=\frac{1}{3}(2\mathbb{H}M_{2n+1}+\mathbb{H}M_{2n-1}-\mathbb{H}M_{2n-2}-(2-\mathbf{i}+2\boldsymbol{\varepsilon}+2\mathbf{h})).$

Corollary 2.7. *For $n\geq 0$, we have the following formulas:*

(a) $\displaystyle\sum_{p=0}^{n}\mathbb{H}R_p=\frac{1}{3}(\mathbb{H}R_{n+2}+2\mathbb{H}R_n+\mathbb{H}R_{n-1}+(2-10\mathbf{i}-13\boldsymbol{\varepsilon}-22\mathbf{h})),$

(b) $\displaystyle\sum_{p=0}^{n}\mathbb{H}R_{2p+1}=\frac{1}{3}(2\mathbb{H}R_{2n+2}+\mathbb{H}R_{2n}-\mathbb{H}R_{2n-1}-(8+2\mathbf{i}+11\boldsymbol{\varepsilon}+11\mathbf{h})),$

(c) $\displaystyle\sum_{p=0}^{n}\mathbb{H}R_{2p}=\frac{1}{3}(2\mathbb{H}R_{2n+1}+\mathbb{H}R_{2n-1}-\mathbb{H}R_{2n-2}+(10-8\mathbf{i}-2\boldsymbol{\varepsilon}-11\mathbf{h})).$

3. Some properties of generalized Tetranacci hybrid numbers

In this section we give some properties of generalized Tetranacci hybrid numbers and as special cases, we present some properties of Tetranacci and Tetranacci-Lucas hybrid numbers.

Note that

$$\widehat{\alpha}+\widehat{\beta}+\widehat{\gamma}+\widehat{\delta}=4+\mathbf{i}+3\boldsymbol{\varepsilon}+7\mathbf{h}.$$

For the generalized Tetranacci hybrid number $\mathbb{H}V_n = V_n+V_{n+1}\mathbf{i}+V_{n+2}\boldsymbol{\varepsilon}+V_{n+3}\mathbf{h}$, we list some definitions as follows.

- The scalar part is $S(\mathbf{Z}) = V_n$ and the vector part is $V(\mathbb{H}V_n) = V_{n+1}\mathbf{i} + V_{n+2}\boldsymbol{\varepsilon} + V_{n+3}\mathbf{h}$.
- The conjugate of $\mathbb{H}V_n$, is $\overline{\mathbb{H}V_n} = S(\mathbb{H}V_n)-V(\mathbb{H}V_n) = V_n - V_{n+1}\mathbf{i}-\boldsymbol{\varepsilon}V_{n+2} - V_{n+3}\mathbf{h}$.
- The characteristic number of $\mathbb{H}V_n$ is

$$\begin{aligned}\mathcal{C}(\mathbb{H}V_n) &= \mathbb{H}V_n\overline{\mathbb{H}V_n} = \overline{\mathbb{H}V_n}\mathbb{H}V_n\\ &= V_n^2+(V_{n+1}-V_{n+2})^2-V_{n+2}^2-V_{n+3}^2\\ &= V_n^2+V_{n+1}^2-2V_{n+2}V_{n+1}-V_{n+3}^2.\end{aligned}$$

- The type number of $\mathbb{H}V_n$ is

$$\begin{aligned}\Delta(\mathbb{H}V_n) &= -(V_{n+1}-V_{n+2})^2+V_{n+2}^2+V_{n+3}^2\\ &= -V_{n+1}^2+2V_{n+2}V_{n+1}+V_{n+3}^2.\end{aligned}$$

- The types of the generalized Tetranacci hybrid numbers are

$$\begin{cases}\mathbb{H}V_n \text{ is elliptic} & \text{if } 2V_{n+2}V_{n+1}+V_{n+3}^2 < V_{n+1}^2;\\ \mathbb{H}V_n \text{ is hyperbolic} & \text{if } 2V_{n+2}V_{n+1}+V_{n+3}^2 > V_{n+1}^2;\\ \mathbb{H}V_n \text{ is parabolic} & \text{if } 2V_{n+2}V_{n+1}+V_{n+3}^2 = V_{n+1}^2.\end{cases}$$

- The norm of $\mathbb{H}V_n$ is

$$\|\mathbb{H}V_n\| = \sqrt{|\mathcal{C}(\mathbb{H}V_n)|} = \sqrt{\left|V_n^2+V_{n+1}^2-2V_{n+2}V_{n+1}-V_{n+3}^2\right|}.$$

- The inverse of $\mathbb{H}V_n$ is

$$\mathbb{H}V_n^{-1}=\frac{\overline{\mathbb{H}V_n}}{\mathcal{C}(\mathbb{H}V_n)}=\frac{V_n-V_{n+1}\mathbf{i}-\boldsymbol{\varepsilon}V_{n+2}-V_{n+3}\mathbf{h}}{V_n^2+V_{n+1}^2-2V_{n+2}V_{n+1}-V_{n+3}^2}$$

where $\|\mathbb{H}V_n\|\neq 0$.

Using the Binet's formula of the generalized Tetranacci hybrid sequence $\{\mathbb{H}V_n\}$, the following theorem immediately follows.

Theorem 3.1. *For any integer n, we have the following:*

(a) $\mathbb{H}V_n+\mathbb{H}V_{n+1}=A\widehat{\alpha}\alpha^{n-6}(1+\alpha)+B\widehat{\beta}\beta^{n-6}(1+\beta)+C\widehat{\gamma}\gamma^{n-6}(1+\gamma)+D\widehat{\delta}\delta^{n-6}(1+\delta)$,

(b) $\mathbb{H}V_n-\mathbb{H}V_{n+1}=A\widehat{\alpha}\alpha^{n-6}(1-\alpha)+B\widehat{\beta}\beta^{n-6}(1-\beta)+C\widehat{\gamma}\gamma^{n-6}(1-\gamma)+D\widehat{\delta}\delta^{n-6}(1-\delta)$,

where A,B,C and D are as in Corollary 1.3.

As special cases, for Tetranacci hybrid number and Tetranacci-Lucas hybrid number, we obtain

$$\begin{aligned}\mathbb{H}M_n+\mathbb{H}M_{n+1}&=\frac{\alpha^2-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1}+\frac{\beta^2-1}{5\beta-8}\widehat{\beta}\beta^{n-1}\\&+\frac{\gamma^2-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1}+\frac{\delta^2-1}{5\delta-8}\widehat{\delta}\delta^{n-1},\\ \mathbb{H}M_n-\mathbb{H}M_{n+1}&=-\frac{(\alpha-1)^2}{5\alpha-8}\widehat{\alpha}\alpha^{n-1}-\frac{(\beta-1)^2}{5\beta-8}\widehat{\beta}\beta^{n-1}\\&-\frac{(\gamma-1)^2}{5\gamma-8}\widehat{\gamma}\gamma^{n-1}-\frac{(\delta-1)^2}{5\delta-8}\widehat{\delta}\delta^{n-1}\end{aligned}$$

and

$$\begin{aligned}\mathbb{H}R_n+\mathbb{H}R_{n+1}&=\widehat{\alpha}\alpha^n(\alpha+1)+\widehat{\beta}\beta^n(\beta+1)+\widehat{\gamma}\gamma^n(\gamma+1)+\widehat{\delta}\delta^n(\delta+1),\\ \mathbb{H}R_n-\mathbb{H}R_{n+1}&=\widehat{\alpha}\alpha^n(1-\alpha)+\widehat{\beta}\beta^n(1-\beta)+\widehat{\gamma}\gamma^n(1-\gamma)+\widehat{\delta}\delta^n(1-\delta).\end{aligned}$$

From Theorem 1.1, we know that

$$\varphi(a+b\mathbf{i}+c\boldsymbol{\varepsilon}+d\mathbf{h})=\begin{pmatrix}a+c & b-c+d\\ c-b+d & a-c\end{pmatrix}=A.$$

So, for $\mathbb{H}V_n=V_n+V_{n+1}\mathbf{i}+V_{n+2}\boldsymbol{\varepsilon}+V_{n+3}\mathbf{h}$ we have

$$\varphi(\mathbb{H}V_n)=\begin{pmatrix}V_n+V_{n+2} & V_{n+1}-V_{n+2}+V_{n+3}\\ V_{n+2}-V_{n+1}+V_{n+3} & V_n-V_{n+2}\end{pmatrix}$$

and as special cases we obtain

$$\varphi(\mathbb{H}M_n) = \begin{pmatrix} M_n + M_{n+2} & M_{n+1} - M_{n+2} + M_{n+3} \\ M_{n+2} - M_{n+1} + M_{n+3} & M_n - M_{n+2} \end{pmatrix}$$

and

$$\varphi(\mathbb{H}R_n) = \begin{pmatrix} R_n + R_{n+2} & R_{n+1} - R_{n+2} + R_{n+3} \\ R_{n+2} - R_{n+1} + R_{n+3} & R_n - R_{n+2} \end{pmatrix}.$$

As basic examples, for $\mathbb{H}M_1 = 1 + \mathbf{i} + 2\varepsilon + 4\mathbf{h}$ and $\mathbb{H}R_1 = 1 + 3\mathbf{i} + 7 + 15\mathbf{h}$ we obtain

$$\varphi(\mathbb{H}M_1) = \begin{pmatrix} 3 & 3 \\ 5 & -1 \end{pmatrix}$$

and

$$\varphi(\mathbb{H}R_1) = \begin{pmatrix} 8 & 11 \\ 19 & -6 \end{pmatrix}.$$

Note that

$$\varphi(\mathbb{H}V_n)\varphi(\overline{\mathbb{H}V_n}) = (V_n^2 + V_{n+1}^2 - 2V_{n+1}V_{n+2} - V_{n+3}^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From Theorem 1.2, we have the following result.

THEOREM 3.2. *Let A be a 2×2 real matrix corresponding to the generalized Tetranacci hybrid number $\mathbb{H}V_n$. Then the followings hold.*
(a) $\|\mathbb{H}V_n\| = \sqrt{|\det A|}$
(b) $\Delta(\mathbb{H}V_n) = \left(\frac{\operatorname{tr} A}{2}\right)^2 - \det A$
(c) $\mathbb{H}V_n^{-1}$ *exists if and only if* $\det(A) \neq 0$.

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