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Hermite–Hadamard type inequalities for m-convex and (α, m)-convex functions

Serap Özcan[1*]

*Correspondence:
serapozcann@yahoo.com;
serapozcan@klu.edu.tr
[1]Department of Mathematics, Faculty of Sciences and Arts, Kırklareli University, Kırklareli, Turkey

Abstract

In this paper, some new inequalities of the Hermite–Hadamard type for the classes of functions whose derivatives' absolute values are m-convex and (α, m)-convex are obtained. The results obtained in this work extend and improve the corresponding ones in the literature. Some applications to special means of real numbers are also given.

MSC: 26D15; 26A51

Keywords: m-convex function; (α, m)-convex function; Hermite–Hadamard type inequalities; Integral inequalities

1 Introduction

Let a real function f be defined on some nonempty interval I of real numbers. The function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f\bigl(tu + (1-t)v\bigr) \le tf(u) + (1-t)f(v)$$

holds for all $u, v \in I$ and $t \in [0, 1]$.

Convexity in connection with integral inequalities is an interesting research area since much attention has been given to studying the concept of convexity and its variant forms in recent years. Some of the most useful inequalities related to the integral mean of a convex function are Hermite–Hadamard's inequality, Jensen's inequality, and Hardy's inequality (see [8, 23–25, 31]). Hermite–Hadamard's inequality provides a necessary and sufficient condition for a function to be convex. This well-known result of Hermite and Hadamard is stated as follows:

If f is a convex function on some nonempty interval I of real numbers and $[u, v] \in I$ with $u < v$, then

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u} \int_u^v f(x)\,dx \le \frac{f(u)+f(v)}{2}. \tag{1}$$

This double inequality may be regarded as a refinement of the concept of convexity, and it follows easily from Jensen's inequality. Recently, a remarkable variety of generalizations and extensions have been considered for the concept of convexity, and related Hermite–Hadamard type integral inequalities have been studied by many researchers (see, for example, [1, 2, 6, 10, 11, 13, 17, 19, 21, 26, 28, 29, 32, 33] and the references cited therein).

2 Preliminaries

We recall the following well-known results and concepts.

Toader [36] introduced the concept of m-convex functions as follows.

Definition 2.1 ([36]) Let $m \in [0,1]$. The function $f : [0,v] \to \mathbb{R}$ is said to be m-convex if

$$f\big(tx + m(1-t)y\big) \leq tf(x) + m(1-t)f(y)$$

is satisfied for every $x, y \in [0,v]$ and $t \in [0,1]$.

It can be easily seen that for $m = 1$, m-convexity reduces to the classical convexity of functions.

Miheşan [22] defined the concept of (α, m)-convex functions as follows.

Definition 2.2 ([22]) Let $\alpha, m \in [0,1]$. The function $f : [0,v] \to \mathbb{R}$ is said to be (α, m)-convex if

$$f\big(tx + m(1-t)y\big) \leq t^{\alpha} f(x) + m\big(1 - t^{\alpha}\big)f(y)$$

is satisfied for every $x, y \in [0,v]$ and $t \in [0,1]$.

Obviously, (α, m)-convexity reduces to m-convexity for $\alpha = 1$ and classical convexity for $\alpha = m = 1$.

For recent results, improvements and generalizations of the concepts of m-convexity and (α, m)-convexity, please refer to the monographs [3–5, 9, 14, 15, 18, 20, 27, 30, 34, 35, 37, 38].

In [7], Dragomir and Agarwal proved the following result connected with the right part of (1).

Lemma 2.1 ([7]) *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (interior of I) and $u, v \in I^{\circ}$ with $u < v$. If $f' \in L[a,b]$, then*

$$\frac{f(u) + f(v)}{2} - \frac{1}{v-u} \int_u^v f(x)\,dx = \frac{v-u}{2} \int_0^1 (1-2t) f'\big(tu + (1-t)v\big)\,dt.$$

Bakula et al. [4] established the following result by using Lemma 2.1 and Hölder's integral inequality.

Theorem 2.1 *Suppose that I is an open real interval such that $[0,\infty) \subset I$, and let $0 \leq u < v < \infty$. Consider the differentiable function $f : I \to \mathbb{R}$ on I such that $f' \in L[u,v]$. If $|f'|^q$ is an* *m-convex function on* $[u,v]$ *for some* $m \in (0,1]$ *and* $q \geq 1$, *then*

$$\left|\frac{f(u)+f(v)}{2} - \frac{1}{v-u}\int_u^v f(x)\,dx\right| \leq \frac{v-u}{4}\min\left\{\left(\frac{|f'(u)|^q + m|f'(\frac{v}{m})|^q}{2}\right)^{\frac{1}{q}}, \left(\frac{m|f'(\frac{u}{m})|^q + |f'(v)|^q}{2}\right)^{\frac{1}{q}}\right\}.$$

İşcan [12] obtained the following integral inequality which gives better results than the classical Hölder integral inequality.

Theorem 2.2 (Hölder–İşcan integral inequality) *Let* f *and* g *be two real functions defined on* $[u,v]$. *If* $|f|^p$ *and* $|g|^q$ *are integrable functions on* $[u,v]$ *for* $p > 1$ *and* $1/p + 1/q = 1$, *then*

$$\begin{aligned}\int_u^v |f(x)g(x)|\,dx &\leq \frac{1}{v-u}\left\{\left(\int_u^v (v-x)|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v (v-x)|g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ &\quad \left. + \left(\int_u^v (x-u)|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v (x-u)|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \\ &\leq \left(\int_u^v |f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_u^v |g(x)|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

İşcan [12] proved the following Hermite–Hadamard type inequality by using Lemma 2.1 and the Hölder–İşcan integral inequality.

Theorem 2.3 *Suppose that* $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable function on* I° *and* $u, v \in I^\circ$ *with* $u < v$. *If* $|f'|^q$ *is a convex function on* $[u,v]$, *then*

$$\left|\frac{f(u)+f(v)}{2} - \frac{1}{v-u}\int_u^v f(x)\,dx\right| \leq \frac{v-u}{4(p+1)^{\frac{1}{p}}}\left\{\left(\frac{2|f'(u)|^q + |f'(v)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{|f'(u)|^q + 2|f'(v)|^q}{3}\right)^{\frac{1}{q}}\right\}. \tag{2}$$

In [16], a different representation of the Hölder–İşcan integral inequality was given as follows.

Theorem 2.4 (Improved power-mean integral inequality) *Let* f *and* g *be two real functions defined on* $[u,v]$. *If* $|f|$, $|f||g|^q$ *are integrable functions on* $[u,v]$ *for* $q \geq 1$, *then*

$$\begin{aligned}&\int_u^v |f(x)g(x)|\,dx \\ &\quad\leq \frac{1}{v-u}\left\{\left(\int_u^v (v-x)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v (v-x)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right. \\ &\qquad\left. + \left(\int_u^v (x-u)|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v (x-u)|f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}\right\} \\ &\quad\leq \left(\int_u^v |f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_u^v |f(x)||g(x)|^q\,dx\right)^{\frac{1}{q}}.\end{aligned}$$

3 Main results

Now we are in a position to establish some new Hermite–Hadamard type inequalities for the classes of m-convex and (α, m)-convex functions.

Theorem 3.1 *Suppose that I is an open real interval such that $[0,\infty) \subset I$, and let $0 \le u < v < \infty$. Consider the differentiable function $f : I \to \mathbb{R}$ on I such that $f' \in L[u,v]$. If $|f'|^q$ is an m-convex function on $[u,v]$ for some $m \in (0,1]$ and $q > 1$, $q = \frac{p}{p-1}$, then*

$$\left|\frac{f(u)+f(v)}{2} - \frac{1}{v-u}\int_u^v f(x)\,dx\right| \le \frac{v-u}{4(p+1)^{\frac{1}{p}}}\left(\lambda_1^{\frac{1}{q}} + \lambda_2^{\frac{1}{q}}\right), \tag{3}$$

where

$$\lambda_1 = \min\left\{\frac{2|f'(u)|^q + m|f'(\frac{v}{m})|^q}{3}, \frac{2m|f'(\frac{u}{m})|^q + |f'(v)|^q}{3}\right\},$$
$$\lambda_2 = \min\left\{\frac{|f'(u)|^q + 2m|f'(\frac{v}{m})|^q}{3}, \frac{m|f'(\frac{u}{m})|^q + 2|f'(v)|^q}{3}\right\}.$$

Proof From Lemma 2.1 and the Hölder–İşcan integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(u)+f(v)}{2} - \frac{1}{v-u}\int_u^v f(x)\,dx\right| \\
&\quad\le \frac{v-u}{2}\int_0^1 |1-2t|\left|f'\bigl(tu+(1-t)v\bigr)\right|dt \\
&\quad\le \frac{v-u}{2}\left\{\left(\int_0^1 (1-t)|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)\left|f'\bigl(tu+(1-t)v\bigr)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\qquad\left. + \left(\int_0^1 t|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\bigl(tu+(1-t)v\bigr)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

From m-convexity of $|f'|^q$ on $[u,v]$ for all $t \in [0,1]$ we have

$$\begin{aligned}
\int_0^1 t\left|f'\bigl(tu+(1-t)v\bigr)\right|^q dt &= \int_0^1 t\left|f'\left(tu + m(1-t)\frac{v}{m}\right)\right|^q dt \\
&\le \frac{2|f'(u)|^q + m|f'(\frac{v}{m})|^q}{6},
\end{aligned}$$

and analogously

$$\begin{aligned}
\int_0^1 t\left|f'\bigl(tu+(1-t)v\bigr)\right|^q dt &= \int_0^1 t\left|f'\left(mt\frac{u}{m} + (1-t)v\right)\right|^q dt \\
&\le \frac{2m|f'(\frac{u}{m})|^q + |f'(v)|^q}{6}.
\end{aligned}$$

So we can write

$$\begin{aligned}
&\int_0^1 t\left|f'\bigl(tu+(1-t)v\bigr)\right|^q dt \\
&\quad\le \min\left\{\frac{2|f'(u)|^q + m|f'(\frac{v}{m})|^q}{6}, \frac{2m|f'(\frac{u}{m})|^q + |f'(v)|^q}{6}\right\}.
\end{aligned} \tag{4}$$

Similarly, we have

$$\begin{aligned}&\int_0^1 (1-t)\left|f'\left(tu+(1-t)v\right)\right|^q\,dt\\&\quad\le \min\left\{\frac{|f'(u)|^q+2m|f'(\frac{v}{m})|^q}{3},\frac{m|f'(\frac{u}{m})|^q+2|f'(v)|^q}{3}\right\}.\end{aligned}\tag{5}$$

Taking into account that

$$\begin{aligned}\int_0^1 t|1-2t|^p\,dt&=\int_0^1(1-t)|1-2t|^p\,dt\\&=\frac{1}{2(p+1)},\end{aligned}\tag{6}$$

we deduce from (4), (5), and (6) inequality (3). □

Remark 3.1 Choosing $m=1$ in inequality (3), we get inequality (2).

Theorem 3.2 *Suppose that I is an open real interval such that $[0,\infty)\subset I$, and let $0\le u<v<\infty$. Consider the differentiable function $f:I\to\mathbb{R}$ on I such that $f'\in L[u,v]$. If $|f'|^q$ is an m-convex function on $[u,v]$ for some $m\in(0,1]$ and $q\ge 1$, then*

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx\right|\le\frac{v-u}{8}\left(\mu_1^{\frac{1}{q}}+\mu_2^{\frac{1}{q}}\right),\tag{7}$$

where

$$\begin{aligned}\mu_1&=\min\left\{\frac{3|f'(u)|^q+m|f'(\frac{v}{m})|^q}{4},\frac{3m|f'(\frac{u}{m})|^q+|f'(v)|^q}{4}\right\},\\\mu_2&=\min\left\{\frac{|f'(u)|^q+3m|f'(\frac{v}{m})|^q}{4},\frac{m|f'(\frac{u}{m})|^q+3|f'(v)|^q}{4}\right\}.\end{aligned}$$

Proof Using Lemma 2.1 and an improved power-mean integral inequality, we have

$$\begin{aligned}&\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx\right|\\&\quad\le\frac{v-u}{2.4^{\frac{1}{p}}}\left\{\left(\int_0^1(1-t)|1-2t|\left|f'\left(tu+(1-t)v\right)\right|^q\,dt\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left(\int_0^1(1-t)|1-2t|\left|f'\left(tu+(1-t)v\right)\right|^q\,dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

By m-convexity of $|f'|^q$ on $[u,v]$ for all $t\in[0,1]$ we have

$$\int_0^1 t|1-2t|\left|f'\left(tu+(1-t)v\right)\right|^q\,dt\le\frac{3|f'(u)|^q+m|f'(\frac{v}{m})|^q}{16},$$

and analogously

$$\int_0^1 t|1-2t|\left|f'\left(tu+(1-t)v\right)\right|^q\,dt\le\frac{3m|f'(\frac{u}{m})|^q+|f'(v)|^q}{16}.$$

So we obtain

$$\int_0^1 t|1-2t|\left|f'\big(tu+(1-t)v\big)\right|^q\,dt$$
$$\leq \min\left\{\frac{3|f'(u)|^q+m|f'(\frac{v}{m})|^q}{16},\frac{3m|f'(\frac{u}{m})|^q+|f'(v)|^q}{16}\right\}. \tag{8}$$

Similarly, we have

$$\int_0^1 (1-t)|1-2t|\left|f'\big(tu+(1-t)v\big)\right|^q\,dt$$
$$\leq \min\left\{\frac{|f'(u)|^q+3m|f'(\frac{v}{m})|^q}{16},\frac{m|f'(\frac{u}{m})|^q+3|f'(v)|^q}{16}\right\}. \tag{9}$$

By using inequalities (8), (9) and the fact that $\int_0^1 t|1-2t|\,dt=\frac{1}{4}$, we get inequality (7). □

Corollary 3.1 *Let the assumptions of Theorem 3.2 be satisfied. If we take $m=1$, then inequality (7) becomes the following inequality*:

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx\right|$$
$$\leq \frac{v-u}{8}\left\{\left(\frac{3|f'(u)|^q+|f'(v)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(u)|^q+3|f'(v)|^q}{4}\right)^{\frac{1}{q}}\right\}. \tag{10}$$

Theorem 3.3 *Suppose that I is an open real interval such that $[0,\infty)\subset I$, and let $0\leq u<v<\infty$. Consider the differentiable function $f:I\to\mathbb{R}$ on I such that $f'\in L[u,v]$. If $|f'|^q$ is an (α,m)-convex function on $[u,v]$ for some $\alpha,m\in(0,1]$ and $q>1$, $q=\frac{p}{p-1}$, then*

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx\right|$$
$$\leq \frac{v-u}{4(p+1)^{\frac{1}{p}}}\left(\varphi_1^{\frac{1}{q}}+\varphi_2^{\frac{1}{q}}\right)\leq \frac{v-u}{4}\left(\varphi_1^{\frac{1}{q}}+\varphi_2^{\frac{1}{q}}\right), \tag{11}$$

where

$$\varphi_1=\min\left\{\frac{\alpha|f'(v)|^q+2m|f'(\frac{u}{m})|^q}{\alpha+2},\frac{2|f'(u)|^q+m\alpha|f'(\frac{v}{m})|^q}{\alpha+2}\right\},$$
$$\varphi_2=\min\left\{\frac{2|f'(u)|^q+m\alpha(\alpha+3)|f'(\frac{v}{m})|^q}{(\alpha+1)(\alpha+2)},\frac{2m|f'(\frac{u}{m})|^q+\alpha(\alpha+3)|f'(v)|^q}{(\alpha+1)(\alpha+2)}\right\}.$$

Proof Using Lemma 2.1 and the Hölder–İşcan integral inequality, we have

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx\right|$$
$$\leq \frac{v-u}{2}\left\{\left(\int_0^1(1-t)|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'\big(tu+(1-t)v\big)\right|^q\,dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\big(tu+(1-t)v\big)\right|^q\,dt\right)^{\frac{1}{q}}\right\}.$$

By (α,m)-convexity of $|f'|^q$ on $[u,v]$ for all $t\in[0,1]$, we get

$$\int_0^1 t\bigl|f'\bigl(tu+(1-t)v\bigr)\bigr|^q\,dt \leq \min\left\{\frac{2|f'(u)|^q+m\alpha|f'(\frac{v}{m})|^q}{2(\alpha+2)},\frac{2m|f'(\frac{u}{m})|^q+\alpha|f'(v)|^q}{2(\alpha+2)}\right\} \tag{12}$$

and

$$\int_0^1 (1-t)\bigl|f'\bigl(tu+(1-t)v\bigr)\bigr|^q\,dt \leq \min\left\{\frac{2|f'(u)|^q+m\alpha(\alpha+3)|f'(\frac{v}{m})|^q}{2(\alpha+1)(\alpha+2)},\frac{2m|f'(\frac{u}{m})|^q+\alpha(\alpha+3)|f'(v)|^q}{2(\alpha+1)(\alpha+2)}\right\}. \tag{13}$$

The proof of the first inequality in (11) is completed by the combination of inequalities (12) and (13). The proof of the second inequality in (11) is completed using the fact

$$\frac{1}{2}<\left(\frac{1}{p+1}\right)^{\frac{1}{p}}<1$$

for $p>1$. □

Corollary 3.2 *Let the assumptions of Theorem 3.3 be satisfied. If we take $m=1$, then inequality (11) becomes the following inequality*:

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int f(x)\,dx\right| \leq \frac{v-u}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{2|f'(u)|^q+\alpha|f'(v)|^q}{\alpha+2}\right)^{\frac{1}{q}}+\left(\frac{2|f'(u)|^q+\alpha(\alpha+3)|f'(v)|^q}{2(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right].$$

Remark 3.2 Inequality (11) yields the right-hand side of Hermite–Hadamard inequality (3) for $\alpha=1$.

Remark 3.3 Choosing $(\alpha,m)=(1,1)$ in the first part of (11), we get inequality (2).

Theorem 3.4 *Suppose that I is an open real interval such that $[0,\infty)\subset I$, and let $0\leq u<v<\infty$. Consider the differentiable function $f:I\to\mathbb{R}$ on I such that $f'\in L[u,v]$. If $|f'|^q$ is an (α,m)-convex function on $[u,v]$ for some $\alpha,m\in(0,1]$ and $q\geq 1$, then*

$$\left|\frac{f(u)+f(v)}{2}-\frac{1}{v-u}\int_u^v f(x)\,dx\right|\leq\frac{v-u}{2.4^{\frac{1}{p}}}\bigl(\tau_1^{\frac{1}{q}}+\tau_2^{\frac{1}{q}}\bigr), \tag{14}$$

where

$$\tau_1=\min\left\{\kappa_1\bigl|f'(u)\bigr|^q+m\kappa_2\left|f'\left(\frac{v}{m}\right)\right|^q,m\kappa_1\left|f'\left(\frac{u}{m}\right)\right|^q+\kappa_2\bigl|f'(v)\bigr|^q\right\},$$
$$\tau_2=\min\left\{\kappa_1^*\bigl|f'(u)\bigr|^q+m\kappa_2^*\left|f'\left(\frac{v}{m}\right)\right|^q,m\kappa_1^*\left|f'\left(\frac{u}{m}\right)\right|^q+\kappa_2^*\bigl|f'(v)\bigr|^q\right\}$$

such that

$$\kappa_1 = \frac{1}{(\alpha+2)(\alpha+3)}\left[\alpha+1+\left(\frac{1}{2}\right)^{\alpha+1}\right], \qquad \kappa_2 = \frac{1}{4}-\kappa_1,$$
$$\kappa_1^* = \frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}\left[\alpha-1+(\alpha+3)\left(\frac{1}{2}\right)^{\alpha}-(\alpha+1)\left(\frac{1}{2}\right)^{\alpha+1}\right], \qquad \kappa_2^* = \frac{1}{4}-\kappa_1^*.$$

Proof Similar to Theorem 3.2 and using (α, m)-convexity of $|f'|^q$, we get the desired result. □

Remark 3.4 If we take $\alpha = 1$ in Theorem 3.4, inequality (14) reduces to inequality (7) in Theorem 3.2.

Remark 3.5 Choosing $\alpha = 1$ and $m = 1$ in Theorem 3.4, we get inequality (10).

4 Applications to special means

We now consider the applications of our results to the following special means for positive real numbers u and v ($u \neq v$).

(1) The arithmetic mean:

$$A := A(u,v) = \frac{u+v}{2};$$

(2) The logarithmic mean:

$$L := L(u,v) = \frac{v-u}{\ln v - \ln u};$$

(3) The generalized logarithmic mean:

$$L_n := L_n(u,v) = \left[\frac{v^{n+1}-u^{n+1}}{(n+1)(v-u)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1,0\}.$$

Proposition 4.1 *Let $u, v \in \mathbb{R}^+$, $u < v$, $m \in (0,1]$, and $n \in \mathbb{Z}\backslash\{-1,0\}$. Then, for all $q \geq 1$, we have*

$$\begin{aligned}
&\left|A\left(u^n, v^n\right) - L_n^n(u,v)\right| \\
&\quad \leq n.\frac{v-u}{8}\left\{\left(\min\left\{\frac{3|u|^{q(n-1)}+m|\frac{v}{m}|^{q(n-1)}}{4}, \frac{3m|\frac{u}{m}|^{q(n-1)}+m|v|^{q(n-1)}}{4}\right\}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\min\left\{\frac{|u|^{q(n-1)}+3m|\frac{v}{m}|^{q(n-1)}}{4}, \frac{m|\frac{u}{m}|^{q(n-1)}+3|v|^{q(n-1)}}{4}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we choose $m = 1$, we obtain

$$\begin{aligned}
&\left|A\left(u^n, v^n\right) - L_n^n(u,v)\right| \\
&\quad \leq n.\frac{v-u}{8}\left[\left(\frac{3|u|^{q(n-1)}+|v|^{q(n-1)}}{4}\right)^{\frac{1}{q}} + \left(\frac{|u|^{q(n-1)}+3|v|^{q(n-1)}}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof The assertions follow from Theorem 3.2 and Corollary 3.1 applied respectively to the m-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. □

Proposition 4.2 *Let $u, v \in \mathbb{R}^+$, $u < v$, $\alpha, m \in (0,1]$, and $n \in \mathbb{Z}\backslash\{-1,0\}$. Then, for all $q \geq 1$, we have*

$$\begin{aligned}
&\bigl|A\bigl(u^n, v^n\bigr) - L_n^n(u,v)\bigr| \\
&\quad \leq n.\frac{v-u}{4(p+1)^{\frac{1}{p}}}\left\{\left(\min\left\{\frac{2|u|^{q(n-1)} + m\alpha|\frac{v}{m}|^{q(n-1)}}{\alpha+2}, \frac{2m|\frac{u}{m}|^{q(n-1)} + \alpha|v|^{q(n-1)}}{\alpha+2}\right\}\right)^{\frac{1}{q}}\right. \\
&\qquad \left. + \left(\min\left\{\frac{2|u|^{q(n-1)} + m\alpha(\alpha+3)|\frac{v}{m}|^{q(n-1)}}{(\alpha+1)(\alpha+2)}, \frac{2m|\frac{u}{m}|^{q(n-1)} + \alpha(\alpha+3)|v|^{q(n-1)}}{(\alpha+1)(\alpha+2)}\right\}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

If we choose $m = 1$, we obtain

$$\begin{aligned}
&\bigl|A\bigl(u^n, v^n\bigr) - L_n^n(u,v)\bigr| \\
&\quad \leq n.\frac{v-u}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{2|u|^{q(n-1)} + \alpha|v|^{q(n-1)}}{\alpha+2}\right)^{\frac{1}{q}} + \left(\frac{2|u|^{q(n-1)} + \alpha(\alpha+3)|v|^{q(n-1)}}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof The assertions follow from Theorem 3.3 and Corollary 3.2 applied respectively to the (α, m)-convex mapping $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$. □

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