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Research article

An approach to energy and elastic for curves with extended Darboux frame in Minkowski space

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Abstract: In this paper, we construct the energy for the ED-frame field of the first and second kind on an orientable hypersurface in Minkowski space. We obtain the geometric properties of some graphics by way of energy. We apply totally diverse discussion and approach to illustrate bending energy functional for the ED-frame field of the first and second kind. Finally, we have a new relation for angle and energy of the curve on an orientable hypersurface in Minkowski space.

Keywords: energy; ED-frame field; hypersurface; vector fields; bending energy
Mathematics Subject Classification: 53A04

1. Introduction

In applied mathematical physics, Minkowski spacetime can easily be imagined seeing that collaboration among time dimension, Euclidean space into a four-dimensional manifold. This important increased time dimension creates a vital improvement among Minkowski space and Euclidean space. This different space framework assists us to appreciate significantly a few numerical and physical trends. Many mathematical researches have been studied in convex geometry, applied geometry, analysis, the geometry of numbers, and lately theoretical pc technology, optimization, combinatorics, [1–7].

Among the initial strategy on elastica produces legendary results on balance of experiences which comprises the fundamental theory of statics. Additionally, it is viewed that elastica offers a pure solution for the variational issue which offers by the minimizing of bending energy source of the elastic curve. Afterward, the equivalence between the motions of the basic pendulum and important differential formula of elastica was researched. Lately, statistical calculation applied upon the elastica utilized to develop mathematical spline principles.

Related principles designed for energy in curvature centered energy is regarded to be at its first phases in development. A few of the legendary fields and landmark research for this theory may be discovered in mathematical physics, membrane chemistry, computer-aided geometric design and geometric modeling, shell engineering, biology and thin plate [8,9] Also, this topic is commonly studied with some solutions [10–19].

We organize the manuscript by starting to state fundamental definitions and propositions for ED-frame fields and energy. Then, we recall the interpretation of the geometrical meaning of the energy for unit vector fields. Based on these relations we compute the energy of curves defined on an orientable hypersurface in E^4 Finally, we give examples about the particle's energy for different cases by computing their value and drawing their graph.

2. Materials and method

Minkowski space E_1^4 refers to four dimensional Euclidean space with the new Lorentzian metric described as

$$\langle p,u\rangle = -p_1u_1 + \sum p_iu_i \mid_{i=2,3,4}$$

where $p,u \in R^4$. For an arbitrary curve $\boldsymbol{\alpha}: I \subset R \to E_1^4$, $\boldsymbol{\alpha} \in E_1^4$ is named lightlike, timelike or spacelike curve if velocity vector of the curve satisfies $\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\rangle = 0$, $\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\rangle < 0$, $\langle \boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\rangle > 0$ for each $t \in I$, respectively, [20]. Also, $\boldsymbol{\alpha}$ is named unit speed curve if $\|\boldsymbol{\alpha}'(t)\| = 1$. In this study we only consider non-lightlike unit speed curves and use a pseudo orthonormal frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ which is attained by Lorentzian rotation on Frenet frame.

$$\begin{aligned}
\mathbf{t}' &= \varepsilon_n k_1 \mathbf{n} \\
\mathbf{n}' &= -\varepsilon_t k_1 \mathbf{t} + \varepsilon_{b_1} k_2 \mathbf{b}_1 \\
\mathbf{b}_1' &= -\varepsilon_n k_2 \mathbf{n} - \varepsilon_t \varepsilon_n \varepsilon_{b_1} k_3 \mathbf{b}_2 \\
\mathbf{b}_2' &= -\varepsilon_{b_1} k_3 \mathbf{b}_1,
\end{aligned}$$

where

$$\varepsilon_t = \langle \mathbf{t}, \mathbf{t}\rangle, \varepsilon_n = \langle \mathbf{n}, \mathbf{n}\rangle, \varepsilon_{b_1} = \langle \mathbf{b}_1, \mathbf{b}_1\rangle, \varepsilon_{b_2} = \langle \mathbf{b}_2, \mathbf{b}_2\rangle.$$

Let M be an orientable hypersurface oriented by unit normal vector field $\mathbf{N}$ in Minkowski space and β be a regular curve on M. putting

$$\mathbf{T}(s) = \beta'(s), \mathbf{N}(s) = \mathbf{N}(\beta(s)).$$

Therefore, we obtain the differential equations of ED-frame fields [21]:

Case 1:

$$
\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \varepsilon_2\kappa_g\mathbf{E}+\varepsilon_4\kappa_n\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{E} &= -\varepsilon_1\kappa_g\mathbf{T}+\varepsilon_3\widetilde{\kappa}_g\mathbf{D}+\varepsilon_4\tau_g\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{D} &= -\varepsilon_2\widetilde{\kappa}_g\mathbf{E}+\varepsilon_4\widetilde{\tau}_g\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{N} &= -\varepsilon_1\kappa_n\mathbf{T}-\varepsilon_2\tau_g\mathbf{E}-\varepsilon_3\widetilde{\tau}_g\mathbf{D}.
\end{aligned}
$$

Case 2:

$$
\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \varepsilon_4\kappa_n\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{E} &= \varepsilon_3\widetilde{\kappa}_g\mathbf{D}+\varepsilon_4\tau_g\mathbf{N},\\
\nabla_{\mathbf{T}}\mathbf{D} &= -\varepsilon_2\widetilde{\kappa}_g\mathbf{E},\\
\nabla_{\mathbf{T}}\mathbf{N} &= -\varepsilon_1\kappa_n\mathbf{T}-\varepsilon_2\tau_g\mathbf{E},
\end{aligned}
$$

where

$$\varepsilon_1=\langle\mathbf{T},\mathbf{T}\rangle,\varepsilon_2=\langle\mathbf{E},\mathbf{E}\rangle,\varepsilon_3=\langle\mathbf{D},\mathbf{D}\rangle,\varepsilon_4=\langle\mathbf{N},\mathbf{N}\rangle.$$

3. Energy with Sasaki metric

For two Riemannian manifolds (M,ρ) and $(N,\widetilde{h})$ the energy of a differentiable map $f:(M,\rho)\to(N,\widetilde{h})$ can be defined as

$$\varepsilon nergy(f)=\frac{1}{2}\int_M\sum_{a=1}^{n}\widetilde{h}(df(e_a),df(e_a))v,$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [22,23].

Let $Q:T(T^1M)\to T^1M$ be the connection map. Then following two conditions hold:

i) $\omega\circ Q=\omega\circ d\omega$ and $\omega\circ Q=\omega\circ\widetilde{\omega}$, where $\widetilde{\omega}:T(T^1M)\to T^1M$ is the tangent bundle projection;
ii) for $\rho\in T_xM$ and a section $\xi:M\to T^1M$; we have

$$Q(d\xi(\rho))=\nabla_\rho\xi,$$

where ∇ is the Levi-Civita covariant derivative [22].

Also, for $\varsigma_1,\varsigma_2\in T_\xi(T^1M)$, we define

$$\rho_S(\varsigma_1,\varsigma_2)=\rho(d\omega(\varsigma_1),d\omega(\varsigma_2))+\rho(Q(\varsigma_1),Q(\varsigma_2)).$$

This yields a Riemannian metric on TM : As we know ρ_S is called the Sasaki metric that also makes the projection $\omega:\ T^1M\to M$ a Riemannian submersion.

4. Energy of ED-frame fields

Firstly, Euler elastica is known as

$$H_B = \frac{1}{2}\int_\beta \left|\nabla_\mathbf{T}\mathbf{T}\right| ds.$$

♠ **Case 1:**

Theorem 1. *Energy of ED-frame field of first kind with Sasaki metric are given by*

$$\begin{aligned}
\varepsilon nergy(\mathbf{T}) &= \frac{1}{2}\int_\beta (1+\varepsilon_2\kappa_g^2+\varepsilon_4\kappa_n^2)ds,\\
\varepsilon nergy(\mathbf{E}) &= \frac{1}{2}\int_\beta (1+\varepsilon_1\kappa_g^2+\varepsilon_3\widetilde{\kappa}_g^2+\varepsilon_4\tau_g^2)ds,\\
\varepsilon nergy(\mathbf{D}) &= \frac{1}{2}\int_\beta (1+\varepsilon_2\widetilde{\kappa}_g^2+\varepsilon_4\widetilde{\tau}_g^2)ds,\\
\varepsilon nergy(\mathbf{N}) &= \frac{1}{2}\int_\beta (1+\varepsilon_1\kappa_n^2+\varepsilon_2\tau_g^2+\varepsilon_3\widetilde{\tau}_g^2)ds.
\end{aligned}$$

Putting

$$\mathbf{X} = \pi_1\mathbf{T}+\pi_2\mathbf{E}+\pi_3\mathbf{D}+\pi_4\mathbf{N}.$$

From ED-frame field of first kind, we obtain

$$\begin{aligned}
\nabla_\mathbf{T}\mathbf{X} &= (\pi_1'-\varepsilon_1\kappa_g\pi_2-\varepsilon_1\kappa_n\pi_4)\mathbf{T}+(\pi_2'-\varepsilon_2\widetilde{\kappa}_g\pi_3+\varepsilon_2\pi_1\kappa_g-\varepsilon_2\tau_g\pi_4)\mathbf{E}\\
&\quad+(\pi_3'-\varepsilon_3\pi_4\widetilde{\tau}_g+\varepsilon_3\widetilde{\kappa}_g\pi_2)\mathbf{D}+(\pi_4'+\varepsilon_4\widetilde{\tau}_g\pi_3+\varepsilon_4\pi_1\kappa_n+\varepsilon_4\tau_g\pi_2)\mathbf{N}.
\end{aligned}$$

Theorem 2. *Energy of field* $\mathbf{X}$ *with ED-frame field of first kind is presented*

$$\begin{aligned}
\varepsilon nergy(\mathbf{X}) &= \frac{1}{2}\int_\beta (\varepsilon_1+\varepsilon_1(\pi_1'-\varepsilon_1\kappa_g\pi_2-\varepsilon_1\kappa_n\pi_4)^2+\varepsilon_2(\pi_2'-\varepsilon_2\widetilde{\kappa}_g\pi_3+\varepsilon_2\pi_1\kappa_g-\varepsilon_2\tau_g\pi_4)^2\\
&\quad+\varepsilon_3(\pi_3'-\varepsilon_3\pi_4\widetilde{\tau}_g+\varepsilon_3\widetilde{\kappa}_g\pi_2)^2+\varepsilon_4(\pi_4'+\varepsilon_4\widetilde{\tau}_g\pi_3+\varepsilon_4\pi_1\kappa_n+\varepsilon_4\tau_g\pi_2)^2)ds.
\end{aligned}$$

Proof. Since $\mathbf{T}$ is a section, we also get

$$d(\omega)\circ d(\mathbf{e}_{(0)}^\mu) = d(\omega\circ\mathbf{e}_{(0)}^\mu) = d(id_C) = id_{TC}.$$

Then, it is easy to see that

$$Q(\mathbf{T}(\mathbf{X})) = \nabla_\mathbf{T}\mathbf{X}.$$

Since

$$\begin{aligned}
Q(\mathbf{T}(\mathbf{X})) &= (\pi_1'-\varepsilon_1\kappa_g\pi_2-\varepsilon_1\kappa_n\pi_4)\mathbf{T}+(\pi_2'-\varepsilon_2\widetilde{\kappa}_g\pi_3+\varepsilon_2\pi_1\kappa_g-\varepsilon_2\tau_g\pi_4)\mathbf{E}\\
&\quad+(\pi_3'-\varepsilon_3\pi_4\widetilde{\tau}_g+\varepsilon_3\widetilde{\kappa}_g\pi_2)\mathbf{D}+(\pi_4'+\varepsilon_4\widetilde{\tau}_g\pi_3+\varepsilon_4\pi_1\kappa_n+\varepsilon_4\tau_g\pi_2)\mathbf{N}.
\end{aligned}$$

Thus, we find

$$\begin{aligned}
\rho_S(d\mathbf{T}(\mathbf{X}),d\mathbf{T}(\mathbf{X})) &= \rho(\mathbf{T},\mathbf{T})+\rho(\nabla_\mathbf{T}\mathbf{X},\nabla_\mathbf{T}\mathbf{X})\\
&= \varepsilon_1+\varepsilon_1(\pi_1'-\varepsilon_1\kappa_g\pi_2-\varepsilon_1\kappa_n\pi_4)^2+\varepsilon_2(\pi_2'-\varepsilon_2\widetilde{\kappa}_g\pi_3+\varepsilon_2\pi_1\kappa_g-\varepsilon_2\tau_g\pi_4)^2\\
&\quad+\varepsilon_3(\pi_3'-\varepsilon_3\pi_4\widetilde{\tau}_g+\varepsilon_3\widetilde{\kappa}_g\pi_2)^2+\varepsilon_4(\pi_4'+\varepsilon_4\widetilde{\tau}_g\pi_3+\varepsilon_4\pi_1\kappa_n+\varepsilon_4\tau_g\pi_2)^2.
\end{aligned}$$

From the above equation, we have theorem.

From elastica we easily have

$$\varepsilon nergy(\mathbf{T}) - H_B = \frac{\varepsilon_1}{2} s.$$

Using above equation we easily have following condition.

Theorem 3. $\mathbf{X}$ *have fixed energy iff*

$$\begin{aligned}
&\varepsilon_2((\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4)^{'} - \varepsilon_2\tilde{\kappa}_g(\pi_3^{'} - \varepsilon_3\pi_4\tilde{\tau}_g + \varepsilon_3\tilde{\kappa}_g\pi_2) - \varepsilon_2\tau_g(\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 \\
&+ \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2) + \varepsilon_2\kappa_g(\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4))(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4) \\
&+ \varepsilon_1((\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4)^{'} - \varepsilon_1\kappa_n(\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2) - \varepsilon_1\kappa_g(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 \\
&+ \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4))(\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4) + \varepsilon_3((\pi_3^{'} - \varepsilon_3\pi_4\tilde{\tau}_g + \varepsilon_3\tilde{\kappa}_g\pi_2)^{'} - \varepsilon_3\tilde{\tau}_g(\pi_4^{'} \\
&+ \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2) + \varepsilon_3\tilde{\kappa}_g(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4))(\pi_3^{'} - \varepsilon_3\pi_4\tilde{\tau}_g + \varepsilon_3\tilde{\kappa}_g\pi_2) \\
&+ \boldsymbol{\varepsilon}_4((\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2)^{'} + \varepsilon_4\tau_g(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4) + \varepsilon_4\tilde{\tau}_g(\pi_3^{'} \\
&- \varepsilon_3\pi_4\tilde{\tau}_g + \varepsilon_3\tilde{\kappa}_g\pi_2) + \varepsilon_4\kappa_n(\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4))(\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2) = 0.
\end{aligned}$$

Proof. Firstly, we have

$$\begin{aligned}
\nabla_{\mathbf{T}}^2\mathbf{X} = {} & ((\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4)^{'} - \varepsilon_1\kappa_n(\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2) \\
&- \varepsilon_1\kappa_g(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4))\mathbf{T} + ((\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4)^{'} \\
&- \varepsilon_2\tilde{\kappa}_g(\pi_3^{'} - \varepsilon_3\pi_4\tilde{\tau}_g + \varepsilon_3\tilde{\kappa}_g\pi_2) - \varepsilon_2\tau_g(\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2) + \varepsilon_2\kappa_g(\pi_1^{'} \\
&- \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4))\mathbf{E} + ((\pi_3^{'} - \varepsilon_3\pi_4\tilde{\tau}_g + \varepsilon_3\tilde{\kappa}_g\pi_2)^{'} - \varepsilon_3\tilde{\tau}_g(\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n \\
&+ \varepsilon_4\tau_g\pi_2) + \varepsilon_3\tilde{\kappa}_g(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4))\mathbf{D} + ((\pi_4^{'} + \varepsilon_4\tilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n \\
&+ \varepsilon_4\tau_g\pi_2)^{'} + \varepsilon_4\tau_g(\pi_2^{'} - \varepsilon_2\tilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4) + \varepsilon_4\tilde{\tau}_g(\pi_3^{'} - \varepsilon_3\pi_4\tilde{\tau}_g \\
&+ \varepsilon_3\tilde{\kappa}_g\pi_2) + \varepsilon_4\kappa_n(\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4))\mathbf{N}.
\end{aligned}$$

With the help of the obtained equation, we express theorem.

Theorem 4. *By ED-frame field of first kind, angle of Frenet vectors can be given*

$$\begin{aligned}
\mathsf{A}(\mathbf{T}) &= \int_0^s \left|(\varepsilon_2\kappa_g^2 + \varepsilon_4\kappa_n^2)\right|^{\frac{1}{2}} ds, \\
\mathsf{A}(\mathbf{E}) &= \int_0^s \left|(\varepsilon_1\kappa_g^2 + \varepsilon_3\tilde{\kappa}_g^2 + \varepsilon_4\tau_g^2)\right|^{\frac{1}{2}} ds, \\
\mathsf{A}(\mathbf{D}) &= \int_0^s (\varepsilon_2\tilde{\kappa}_g^2 + \varepsilon_4\tilde{\tau}_g^2)^{\frac{1}{2}} ds, \\
\mathsf{A}(\mathbf{N}) &= \int_0^s (\varepsilon_1\kappa_n^2 + \varepsilon_2\tau_g^2 + \varepsilon_3\tilde{\tau}_g^2)^{\frac{1}{2}} ds.
\end{aligned}$$

Theorem 5. *Generalization angle of any field* $\mathbf{X}$ *is given by*

$$\mathsf{A}(\mathbf{X}) = \int_0^s (\varepsilon_1(\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4)^2 + \varepsilon_2(\pi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4)^2$$
$$+\varepsilon_3(\pi_3^{'} - \varepsilon_3\pi_4\widetilde{\tau}_g + \varepsilon_3\widetilde{\kappa}_g\pi_2)^2 + \varepsilon_4(\pi_4^{'} + \varepsilon_4\widetilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2)^2)^{\frac{1}{2}}ds.$$

Theorem 6. *If pseudo angle of each of Frenet vectors given above are fixed, then we have following relation:*

$$\varepsilon_1(\pi_1^{'} - \varepsilon_1\kappa_g\pi_2 - \varepsilon_1\kappa_n\pi_4)^2 + \varepsilon_2(\pi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\pi_3 + \varepsilon_2\pi_1\kappa_g - \varepsilon_2\tau_g\pi_4)^2$$
$$+\varepsilon_3(\pi_3^{'} - \varepsilon_3\pi_4\widetilde{\tau}_g + \varepsilon_3\widetilde{\kappa}_g\pi_2)^2 + \varepsilon_4(\pi_4^{'} + \varepsilon_4\widetilde{\tau}_g\pi_3 + \varepsilon_4\pi_1\kappa_n + \varepsilon_4\tau_g\pi_2)^2 = 0$$

♠ **Case 2:**

Theorem 7. *Energy of ED-frame field of second kind with Sasaki metric are given by*

$$\varepsilon nergy(\mathbf{T}) = \frac{1}{2}\int_\beta(\varepsilon_1 + \varepsilon_4\kappa_n^2)ds,$$
$$\varepsilon nergy(\mathbf{E}) = \frac{1}{2}\int_\beta(\varepsilon_1 + \varepsilon_3\widetilde{\kappa}_g^2 + \varepsilon_4\tau_g^2)ds,$$
$$\varepsilon nergy(\mathbf{D}) = \frac{1}{2}\int_\beta(\varepsilon_1 + \varepsilon_2\widetilde{\kappa}_g^2)ds,$$
$$\varepsilon nergy(\mathbf{N}) = \frac{1}{2}\int_\beta(\varepsilon_1 + \varepsilon_1\kappa_n^2 + \varepsilon_2\tau_g^2)ds.$$

Putting

$$\mathbf{X} = \varpi_1\mathbf{T} + \varpi_2\mathbf{E} + \varpi_3\mathbf{D} + \varpi_4\mathbf{N}.$$

Theorem 8. *Generalization energy of any field* $\mathbf{X}$ *is given by*

$$\varepsilon nergy(\mathbf{X}) = \frac{1}{2}\int_\beta(\varepsilon_1 + \varepsilon_1(\varpi_1^{'} - \varepsilon_1\kappa_n\varpi_4)^2 + \varepsilon_2(\varpi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4)^2$$
$$+\varepsilon_3(\varpi_3^{'} + \varepsilon_3\widetilde{\kappa}_g\varpi_2)^2 + \varepsilon_4(\varpi_4^{'} + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2)^2)ds.$$

Proof. From ED-frame field of second kind, we have

$$\mathbf{X}^{'} = (\varpi_1^{'} - \varepsilon_1\kappa_n\varpi_4)\mathbf{T} + (\varpi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4)\mathbf{E}$$
$$+(\varpi_3^{'} + \varepsilon_3\widetilde{\kappa}_g\varpi_2)\mathbf{D} + (\varpi_4^{'} + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2)\mathbf{N}$$

and

$$\mathbf{X}^{''} = ((\varpi_1^{'} - \varepsilon_1\kappa_n\varpi_4) - \varepsilon_1\kappa_n(\varpi_4^{'} + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2))\mathbf{T} + ((\varpi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4)$$
$$-\varepsilon_2\widetilde{\kappa}_g(\varpi_3^{'} + \varepsilon_3\widetilde{\kappa}_g\varpi_2) - \varepsilon_2\tau_g(\varpi_4^{'} + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2))\mathbf{E} + ((\varpi_3^{'} + \varepsilon_3\widetilde{\kappa}_g\varpi_2)$$
$$+\varepsilon_3\widetilde{\kappa}_g(\varpi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4))\mathbf{D} + ((\varpi_4^{'} + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2)$$
$$+\varepsilon_4\tau_g(\varpi_2^{'} - \varepsilon_2\widetilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4) + \varepsilon_4(\varpi_1^{'} - \varepsilon_1\kappa_n\varpi_4)\kappa_n)\mathbf{N}.$$

Thus, we have theorem.

Theorem 9. **X** *have fixed energy with ED-frame field of second kind iff*

$$\varepsilon_1((\varpi_1' - \varepsilon_1\kappa_n\varpi_4) - \varepsilon_1\kappa_n(\varpi_4' + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2))(\varpi_1' - \varepsilon_1\kappa_n\varpi_4) + \varepsilon_2((\varpi_2' - \varepsilon_2\tilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4) - \varepsilon_2\tilde{\kappa}_g(\varpi_3' + \varepsilon_3\tilde{\kappa}_g\varpi_2) - \varepsilon_2\tau_g(\varpi_4' + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2))(\varpi_2' - \varepsilon_2\tilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4) + \varepsilon_3((\varpi_3' + \varepsilon_3\tilde{\kappa}_g\varpi_2) + \varepsilon_3\tilde{\kappa}_g(\varpi_2' - \varepsilon_2\tilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4))(\varpi_3' + \varepsilon_3\tilde{\kappa}_g\varpi_2) + \varepsilon_4((\varpi_4' + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2) + \varepsilon_4\tau_g(\varpi_2' - \varepsilon_2\tilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4) + \varepsilon_4(\varpi_1' - \varepsilon_1\kappa_n\varpi_4)\kappa_n)(\varpi_4' + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2) = 0.$$

Theorem 10. *Angle of Frenet vectors can be respectively given by*

$$\begin{aligned}
\mathrm{A}(\mathbf{T}) &= \int_0^s (\varepsilon_4\kappa_n)\,ds,\\
\mathrm{A}(\mathbf{E}) &= \int_0^s (\varepsilon_3\tilde{\kappa}_g^2 + \varepsilon_4\tau_g^2)^{\frac{1}{2}}\,ds,\\
\mathrm{A}(\mathbf{D}) &= \int_0^s (\varepsilon_2\tilde{\kappa}_g)\,ds,\\
\mathrm{A}(\mathbf{N}) &= \int_0^s (\varepsilon_1\kappa_n^2 + \varepsilon_2\tau_g^2)^{\frac{1}{2}}\,ds.
\end{aligned}$$

Theorem 11. *Generalization angle of any field* **X** *is given by*

$$\mathrm{A}(\mathbf{X}) = \int_0^s (\varepsilon_1(\varpi_1' - \varepsilon_1\kappa_n\varpi_4)^2 + \varepsilon_2(\varpi_2' - \varepsilon_2\tilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4)^2 + \varepsilon_3(\varpi_3' + \varepsilon_3\tilde{\kappa}_g\varpi_2)^2 + \varepsilon_4(\varpi_4' + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2)^2)^{\frac{1}{2}}\,ds.$$

Theorem 12. *If angle of each of Frenet vectors are fixed, then we have*

$$\varepsilon_1(\varpi_1' - \varepsilon_1\kappa_n\varpi_4)^2 + \varepsilon_2(\varpi_2' - \varepsilon_2\tilde{\kappa}_g\varpi_3 - \varepsilon_2\tau_g\varpi_4)^2 + \varepsilon_3(\varpi_3' + \varepsilon_3\tilde{\kappa}_g\varpi_2)^2 + \varepsilon_4(\varpi_4' + \varepsilon_4\kappa_n\varpi_1 + \varepsilon_4\tau_g\varpi_2)^2 = 0.$$

5. Application

Let M be an orientable hypersurface oriented by the unit normal vector field **N** in E_1^4. If β is timelike helix in E_1^4, then we have following applications with angle and energy (Figures 1–5).

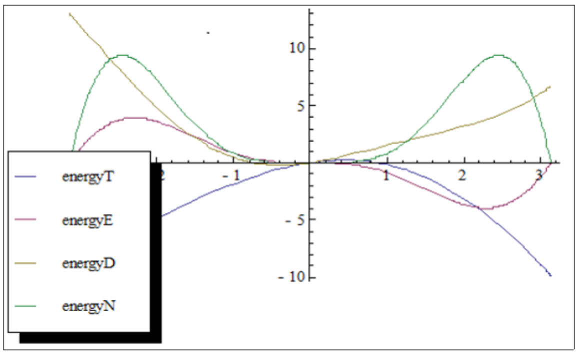

Fig. 1. Plots of energy with ED-frame field of first kind

Figure 1. Plots of energy with ED-fame field of first kind.

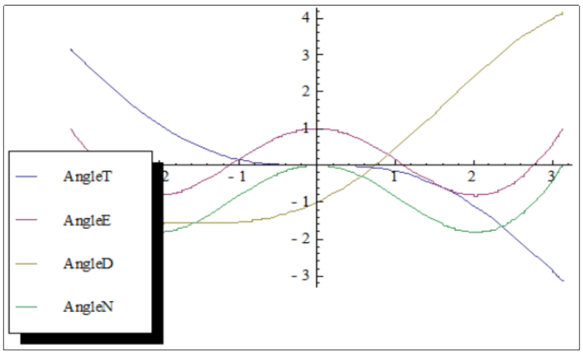

Fig. 2. Plots of angle with ED-frame field of first kind

Figure 2. Plots of angle with ED-frame field of first kind.

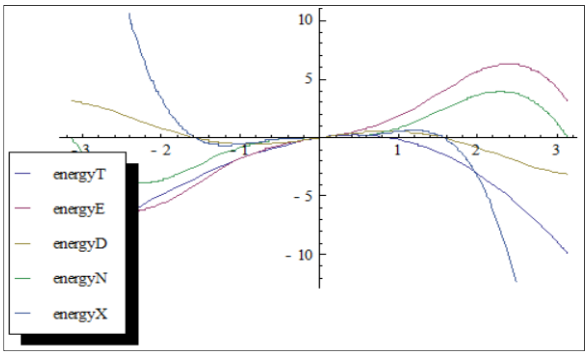

Fig. 3. Plots of energy with X, ED-frame field of first kind

Figure 3. Plots of angle with X, ED-frame field of first kind.

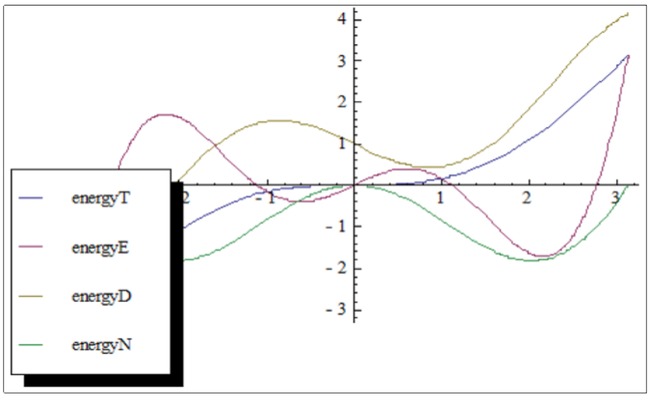

Fig. 4. Plots of energy with ED-frame field of second kind

Figure 4. Plots of energy with ED-frame field of second bind.

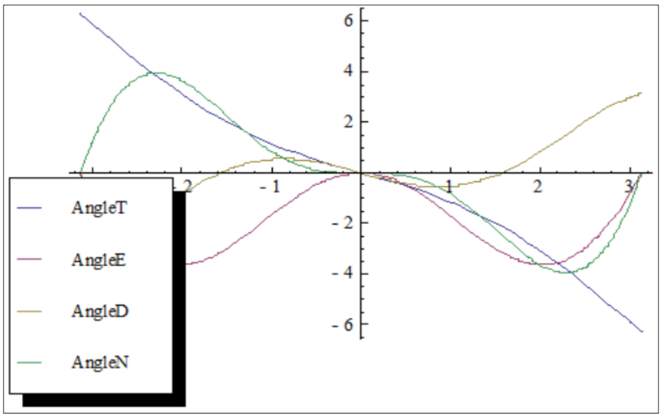

Fig. 5. Plots of angle with ED-frame field of second kind

Figure 5. Plots of angle with ED-frame field of second bind.

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Conflict of interest

The authors declare no conflict of interest.

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