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## ON THE GLOBAL ASYMPTOTIC STABILITY OF A TWO DIMENSIONAL SYSTEM OF DIFFERENCE EQUATIONS WITH QUADRATIC TERMS

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ABSTRACT. In this paper, we study the global asymptotically stability of following system of difference equations with quadratic terms:

$$x_{n+1} = A + B \frac{y_n}{y_{n-1}^2}, y_{n+1} = A + B \frac{x_n}{x_{n-1}^2}$$

where  $A$  and  $B$  are positive numbers and the initial values are positive numbers. We also investigate the rate of convergence and oscillation behaviour of the solutions of related system.

### 1. INTRODUCTION

Over the last two decades, difference equations or their systems have been huge attention among researchers which is mathematician or not. Moreover difference equations or systems have too many applications between many branches of science. For example, in [13] Khan et al studied global dynamics of an one-dimensional discrete-time laser model. Further in [7] Din et al studied stability of a discrete ecological model. There are many examples related to applications of difference equations or systems. Therefore, studies on difference equations are increasing day by day and will continue to increase. Additionally, there are many papers related to our study as follows:

In [24], Yang et al investigated the solutions, stability and asymptotic behavior of the system of the two nonlinear difference equations

$$x_{n+1} = \frac{Ax_n}{1 + y_n^p}, y_{n+1} = \frac{By_n}{1 + x_n^p}.$$

In [8], Elabbasy et al studied the global behaviour of following system of difference equations

$$x_{n+1} = \frac{a_1 x_n}{a_2 + a_3 y_n^r}, y_{n+1} = \frac{b_1 y_n}{b_2 + b_3 x_n^r}.$$

In [3], Bacani et al considered solutions of the following two nonlinear difference equations

$$x_{n+1} = \frac{q}{p + x_n^v}, y_{n+1} = \frac{q}{-p + y_n^v}.$$

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In [11], Hadziabdic et al studied global behaviours of following system of difference equations

$$x_{n+1} = \frac{b_1 x_n^2}{A_1 + y_n^2}, y_{n+1} = \frac{a_2 + c_2 y_n^2}{x_n^2}.$$

In [5], Burgic et al investigated the global stability properties and asymptotic behavior of solutions for the system of difference equations

$$x_{n+1} = \frac{x_n}{a + y_n^2}, y_{n+1} = \frac{y_n}{b + x_n^2}.$$

In [4], Beso et al studied boundedness of solutions of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}.$$

They also investigated global asymptotic stability of related difference equation.

Motivated by difference equations and their systems, we consider the following system of difference equations

$$(1.1) \quad x_{n+1} = A + B \frac{y_n}{y_{n-1}^2}, y_{n+1} = A + B \frac{x_n}{x_{n-1}^2}$$

where  $A$  and  $B$  are positive numbers and the initial values are positive numbers. In this paper we study the stability, global behaviour and rate of convergence of solutions of system (1.1). We also investigate the oscillation behaviour of solutions of related system.

In this here, we give two theorems which are used during this study.

**Theorem 1.1** (Linearized Stability Theorem [16], p.11). *Assume that*

$$X_{n+1} = F(X_n), n = 0, 1, \dots,$$

*is a system of difference equations such that  $\bar{X}$  is a fixed point of  $F$ .*

- (i): *If all eigenvalues of the Jacobian matrix  $B$  about  $\bar{X}$  lie inside the open unit disk  $|\lambda| < 1$ , that is, if all of them have absolute value less than one, then  $\bar{X}$  is locally asymptotically stable.*
- (ii): *If at least one of them has a modulus greater than one, then  $\bar{X}$  is unstable.*

**Theorem 1.2.** [2] *Let  $n \in N_{n_0}^+$  and  $g(n, u, v)$  be a decreasing function in  $u$  and  $v$  for any fixed  $n$ . Suppose that for  $n \geq n_0$ , the inequalities*

$$\begin{aligned} y_{n+1} &\leq g(n, y_n, y_{n-1}) \\ u_{n+1} &\geq g(n, y_n, y_{n-1}) \end{aligned}$$

*hold. Then*

$$y_{n_0-1} \leq u_{n_0-1}, y_{n_0} \leq u_{n_0}$$

*implies that*

$$y_n \leq u_n, n \geq n_0.$$

## 2. LINEARIZED STABILITY OF SYSTEM

First of all, we consider the change of the variables for system (1.1) as follows:

$$t_n = \frac{x_n}{A}, z_n = \frac{y_n}{A}.$$

From this, system (1.1) transform into following system:

$$(2.1) \quad t_{n+1} = 1 + p \frac{z_n}{z_{n-1}^2}, z_{n+1} = 1 + p \frac{t_n}{t_{n-1}^2}$$

where  $p = \frac{B}{A^2} > 0$ . From now on, we study the system (2.1).

**Lemma 2.1.** *Let  $p > 0$ . Unique positive equilibrium point of system (2.1) is*

$$(\bar{t}, \bar{z}) = \left( \frac{1 + \sqrt{1 + 4p}}{2}, \frac{1 + \sqrt{1 + 4p}}{2} \right).$$

Now, we consider a transformation as follows:

$$(t_n, t_{n-1}, z_n, z_{n-1}) \rightarrow (f, f_1, g, g_1)$$

where  $f = 1 + p \frac{z_n}{z_{n-1}^2}$ ,  $f_1 = t_n$ ,  $g = 1 + p \frac{t_n}{t_{n-1}^2}$ ,  $g_1 = z_n$ . Thus we get the jacobian matrix about equilibrium point  $(\bar{t}, \bar{z})$ :

$$B(\bar{t}, \bar{z}) = \begin{pmatrix} 0 & 0 & \frac{p}{\bar{z}^2} & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & 0 & 0 \\ \frac{p}{\bar{t}^2} & \frac{-2p}{\bar{t}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the linearized system of system (2.1) about the unique positive equilibrium point is given by  $X_{N+1} = B(\bar{t}, \bar{z}) X_N$ , where

$$X_N = \begin{pmatrix} t_n \\ t_{n-1} \\ z_n \\ z_{n-1} \end{pmatrix},$$

$$B(\bar{t}, \bar{z}) = \begin{pmatrix} 0 & 0 & \frac{p}{\bar{z}^2} & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & 0 & 0 \\ \frac{p}{\bar{t}^2} & \frac{-2p}{\bar{t}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, the characteristic equation of  $B(\bar{t}, \bar{z})$  about the unique positive equilibrium point  $(\bar{t}, \bar{z})$  is

$$\lambda^4 - \frac{p^2}{\bar{t}^2 \bar{z}^2} \lambda^2 + \frac{4p^2}{\bar{t}^2 \bar{z}^2} \lambda - \frac{4p^2}{\bar{t}^2 \bar{z}^2} = 0.$$

Due to  $\bar{t} = \bar{z}$ , we can rearrange the characteristic equation such that

$$\lambda^4 - \frac{p^2}{\bar{t}^4} \lambda^2 + \frac{4p^2}{\bar{t}^4} \lambda - \frac{4p^2}{\bar{t}^4} = 0.$$

Therefore, we obtain the four roots of characteristic equation as follows:

$$\begin{aligned}\lambda_1 &= \frac{p + \sqrt{p^2 - 8pt^2}}{2t^2}, \\ \lambda_2 &= \frac{p - \sqrt{p^2 - 8pt^2}}{2t^2}, \\ \lambda_3 &= \frac{-p + \sqrt{p^2 + 8pt^2}}{2t^2}, \\ \lambda_4 &= \frac{-p - \sqrt{p^2 + 8pt^2}}{2t^2}.\end{aligned}$$

Now, we calculate  $\bar{t}^2$  and write in  $\lambda_1$ . Then we have

$$\begin{aligned}\lambda_1 &= \frac{p + \sqrt{p^2 - 4p(1 + 2p + \sqrt{4p + 1})}}{1 + 2p + \sqrt{4p + 1}} \\ &= \frac{p + \sqrt{-7p^2 - 4p - 4p\sqrt{1 + 4p}}}{1 + 2p + \sqrt{4p + 1}} \\ &= \frac{p + \sqrt{7p^2 + 4p + 4p\sqrt{1 + 4p}i}}{1 + 2p + \sqrt{4p + 1}}.\end{aligned}$$

Thus straightforward calculations show that

$$|\lambda_1| = \frac{2\sqrt{2p}}{1 + \sqrt{1 + 4p}}.$$

Additionally, we obtain similarly calculations that

$$|\lambda_2| = \frac{2\sqrt{2p}}{1 + \sqrt{1 + 4p}}.$$

On the other hand, we consider  $\lambda_3$  as follows:

$$\begin{aligned}\lambda_3 &= \frac{-p + \sqrt{9p^2 + 4p + 4p\sqrt{4p + 1}}}{1 + 2p + \sqrt{4p + 1}} \\ &= \frac{-p + \sqrt{(3p + \sqrt{1 + 4p})^2 - 1 - 2p\sqrt{4p + 1}}}{1 + 2p + \sqrt{4p + 1}} \\ &< \frac{-p + \sqrt{(3p + \sqrt{1 + 4p})^2}}{1 + 2p + \sqrt{4p + 1}} \\ &= \frac{2p + \sqrt{1 + 4p}}{1 + 2p + \sqrt{4p + 1}} < 1.\end{aligned}$$

Moreover we clearly see that  $\lambda_3 > 0$ . So  $0 < \lambda_3 < 1$  for all  $p > 0$ . Similar calculations we have that  $-1 < \lambda_4 < 0$  for all  $p > 0$ .

**Theorem 2.2.** *Suppose that  $p > 0$ . Then the following cases hold for system (2.1):*

- i.:** *If  $p < 2$  then the equilibrium point of system (2.1) is locally asymptotically stable.*
- ii.:** *If  $p = 2$  then the equilibrium point of system (2.1) is a non-hyperbolic equilibrium.*
- iii.:** *If  $p > 2$  then the equilibrium point of system (2.1) is a repeller.*

*Proof.* Firstly we know that  $|\lambda_3|, |\lambda_4| < 1$  for all  $p > 0$ . Now we consider

$$|\lambda_1| = |\lambda_2| = \frac{2\sqrt{2p}}{1 + \sqrt{1 + 4p}}.$$

If the equilibrium point of system (2.1) is locally asymptotically stable, then all roots of characteristic equation must lie the unit disk. Therefore, we must show that  $|\lambda_1|, |\lambda_2| < 1$ . Hence

$$|\lambda_1| = |\lambda_2| = \frac{2\sqrt{2p}}{1 + \sqrt{1 + 4p}} < 1.$$

Thus, we have  $2\sqrt{2p} < 1 + \sqrt{1 + 4p}$ . From this, we obtain that  $p < 2$ . The proofs of other cases can be obtained in a similar way.  $\square$

### 3. AN OSCILLATION RESULT OF SOLUTIONS OF SYSTEM (2.1)

In this here, we investigate the oscillation behaviour of solutions of system(2.1).

**Theorem 3.1.** *Let  $\{(t_n, z_n)\}$  be a positive solution of system (2.1) and  $p > 0$ . Then for any  $n \geq 0$  the following cases are true.*

**i.:** *If  $t_{n+1}, z_n < \bar{t} = \bar{z} < t_n, z_{n+1}$  then*

$$\begin{aligned} (t_{n+2k-1})_{k=1}^{\infty} &< \bar{t} < (t_{n+2k})_{k=1}^{\infty}, \\ (z_{n+2k})_{k=1}^{\infty} &< \bar{z} < (z_{n+2k-1})_{k=1}^{\infty}. \end{aligned}$$

**ii.:** *If  $t_n, z_{n+1} < \bar{t} = \bar{z} < t_{n+1}, z_n$  then*

$$\begin{aligned} (t_{n+2k})_{k=1}^{\infty} &< \bar{t} < (t_{n+2k-1})_{k=1}^{\infty}, \\ (z_{n+2k-1})_{k=1}^{\infty} &< \bar{z} < (z_{n+2k})_{k=1}^{\infty}. \end{aligned}$$

*Proof.* Firstly we consider case (i). Assume that  $t_{n+1}, z_n < \bar{t} = \bar{z} < t_n, z_{n+1}$ . Then we obtain that

$$\begin{aligned} t_{n+2} &= 1 + p \frac{z_{n+1}}{z_n^2} > 1 + p \frac{\bar{z}}{\bar{z}^2} = \bar{z} = \bar{t}, \\ z_{n+2} &= 1 + p \frac{t_{n+1}}{t_n^2} < 1 + p \frac{\bar{t}}{\bar{t}^2} = \bar{t} = \bar{z}, \\ t_{n+3} &< \bar{t}, z_{n+3} > \bar{z}, t_{n+4} > \bar{t}, z_{n+4} < \bar{z}. \end{aligned}$$

Therefore we have by using induction

$$\begin{aligned} t_n, t_{n+2}, \dots, t_{n+2k}, \dots &< \bar{t} < t_{n+1}, t_{n+3}, \dots, t_{n+2k-1}, \dots, \\ z_{n+1}, z_{n+3}, \dots, z_{n+2k-1}, \dots &< \bar{z} < z_n, z_{n+2}, \dots, z_{n+2k}, \dots \end{aligned}$$

Thus the proof of (i) is completed as desired. The proof of (ii) is similar to proof of (i).  $\square$

### 4. BOUNDEDNESS OF SYSTEM (2.1)

**Lemma 4.1.** *Let  $\{(t_n, z_n)\}$  be a positive solution of system (2.1) and  $p > 0$ . Then  $t_n > 1$  and  $z_n > 1$  for  $n \geq 1$ .*

*Proof.* Let  $\{(t_n, z_n)\}$  be a positive solution of system (2.1). Then we have from system (2.1):

$$\begin{aligned}t_1 &= 1 + p \frac{z_0}{z_{-1}^2} > 1, \\z_1 &= 1 + p \frac{t_0}{t_{-1}^2} > 1.\end{aligned}$$

Therefore, we obtain by induction

$$\begin{aligned}t_{n+1} &= 1 + p \frac{z_n}{z_{n-1}^2} > 1, \\z_{n+1} &= 1 + p \frac{t_n}{t_{n-1}^2} > 1.\end{aligned}$$

So, the proof of lemma is completed.  $\square$

**Theorem 4.2.** *If  $0 < p < 1$  then every solution of system (2.1) is bounded.*

*Proof.* Firstly we have from system (2.1)  $t_n > 1$  and  $z_n > 1$  for  $n \geq 1$  and  $p > 0$ . Moreover, every solution of system (2.1) satisfies

$$(4.1) \quad t_{n+1} \leq 1 + p + p^2 t_{n-1}, n \geq 1,$$

which due to Theorem 1.2, means that  $t_n \leq q_n$ ,  $n = 0, 1, \dots$ , where  $\{u_n\}$  satisfy

$$(4.2) \quad u_{n+1} = 1 + p + p^2 u_{n-1}, n \geq 1,$$

such that

$$u_s = t_s, u_{s+1} = t_{s+1}, s \in \{-1, 0, 1, \dots\}, n \geq s.$$

Hence the solution  $u_n$  of the difference equation (4.2) is

$$(4.3) \quad u_n = \frac{1}{1-p} + p^n C_1 + (-p)^n C_2.$$

Moreover, we have from (4.2)

$$u_{n+1} = 1 + p + p^2 u_{n-1} \Rightarrow \lambda^2 - p^2 = 0 \Rightarrow \lambda_{1,2} = \pm p.$$

From this, the homogeneous solution of difference equation (4.2) is

$$u_h = p^n C_1 + (-p)^n C_2.$$

In addition, from (4.2), the equilibrium solution of difference equation (4.2) is

$$\bar{u} = 1 + p + p^2 \bar{u} \Rightarrow \bar{u} = \frac{1}{1-p}.$$

Additionally, relations (4.1) and (4.3) imply that

$$t_{n+1} - u_{n+1} \leq p^2 (t_{n-1} - u_{s-1}), n > s, p \in (0, 1).$$

Therefore we have

$$(4.4) \quad t_n \leq u_n, n > s.$$

Hence, we obtain from (4.3), (4.4) and Lemma 4.1,

$$1 < t_n \leq \frac{1}{1-p} + p^n C_1 + (-p)^n C_2 = M_1,$$

where

$$\begin{aligned} C_1 &= \frac{1}{2p} \left( pt_0 + t_1 - \frac{1+p}{1-p} \right), \\ C_2 &= \frac{1}{2p} (pt_0 - t_1 + 1). \end{aligned}$$

Similarly we can write that

$$1 < z_n \leq \frac{1}{1-p} + p^n C_3 + (-p)^n C_4 = M_2,$$

where

$$\begin{aligned} C_3 &= \frac{1}{2p} \left( pz_0 + z_1 - \frac{1+p}{1-p} \right), \\ C_4 &= \frac{1}{2p} (pz_0 - z_1 + 1). \end{aligned}$$

□

## 5. CONVERGENCE RESULTS OF SOLUTIONS OF SYSTEM (2.1)

**Theorem 5.1.** *If  $t_n \geq \bar{t}$  and  $z_n \geq \bar{z}$  (resp.,  $t_n \leq \bar{t}$  and  $z_n \leq \bar{z}$ ) for  $n \geq s$  and  $s \in \{-1, 0, \dots\}$  then the solution  $\{(t_n, z_n)\}$  of system (2.1) tends to equilibrium point  $(\bar{t}, \bar{z})$  as  $n \rightarrow \infty$ .*

*Proof.* Let  $\{(t_n, z_n)\}$  be a positive solution of system (2.1) such that

$$(5.1) \quad t_n \geq \bar{t}, z_n \geq \bar{z}, n \geq s,$$

where  $s \in \{-1, 0, \dots\}$ . Hence, we obtain from (5.1), system (2.1) and Lemma 4.1:

$$(5.2) \quad t_{n+1} \leq 1 + p + p^2 t_{n-1}.$$

Now we set

$$(5.3) \quad u_{n+1} = 1 + p + p^2 u_{n-1},$$

such that

$$(5.4) \quad u_s = t_s, u_{s+1} = t_{s+1}, s \in \{-1, 0, \dots\}, n \geq s.$$

Therefore, we get from the solution of the difference equation (5.3):

$$(5.5) \quad u_n = \frac{1}{1-p} + p^n C_1 + (-p)^n C_2$$

where  $C_1, C_2$  depend on  $t_s, t_{s+1}$ . Moreover, we have from (5.2) and (5.3):

$$(5.6) \quad t_{n+1} - u_{s+1} \leq p^2 (t_{n-1} - u_{s-1}), n > s.$$

Thus we obtain from (5.4), (5.6) and by induction

$$(5.7) \quad t_n \leq u_n, n \geq s.$$

From (5.1), (5.5) and (5.7), we obtain that

$$\lim_{n \rightarrow \infty} t_n = \bar{t}.$$

Then we similarly obtain that  $\lim_{n \rightarrow \infty} z_n = \bar{z}$ . The proof of the other case of this theorem is similar to this case, so we leave it to readers. □

**Theorem 5.2.** *Assume that  $0 < p < \frac{1}{2}$ . Then the positive equilibrium point of system (2.1) is globally asymptotically stable.*

*Proof.* We have from Theorem 4.2,

$$\begin{aligned} 1 &< l_1 = \lim_{n \rightarrow \infty} t_n \leq M_1, \\ 1 &< l_2 = \lim_{n \rightarrow \infty} z_n \leq M_2, \\ 1 &< U_1 = \lim_{n \rightarrow \infty} t_n \leq M_1, \\ 1 &< U_2 = \lim_{n \rightarrow \infty} z_n \leq M_2. \end{aligned}$$

By system (2.1), we can write

$$\begin{aligned} U_1 &\leq 1 + p \frac{U_2}{l_2^2}, l_1 \geq 1 + p \frac{l_2}{U_2^2}, \\ U_2 &\leq 1 + p \frac{U_1}{l_1^2}, l_2 \geq 1 + p \frac{l_1}{U_1^2}. \end{aligned}$$

Hence we have

$$\begin{aligned} U_1 + p \frac{l_1}{U_1} &\leq U_1 l_2 \leq l_2 + p \frac{U_2}{l_2}, \\ U_2 + p \frac{l_2}{U_2} &\leq U_2 l_1 \leq l_1 + p \frac{U_1}{l_1}. \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} U_1 + p \frac{l_1}{U_1} + U_2 + p \frac{l_2}{U_2} &\leq l_2 + p \frac{U_2}{l_2} + l_1 + p \frac{U_1}{l_1}, \\ U_1 + p \frac{l_1}{U_1} + U_2 + p \frac{l_2}{U_2} - l_2 - p \frac{U_2}{l_2} - l_1 - p \frac{U_1}{l_1} &\leq 0, \\ (U_1 - l_1) \left( 1 - p \left( \frac{1}{l_1} + \frac{1}{U_1} \right) \right) + (U_2 - l_2) \left( 1 - p \left( \frac{1}{l_2} + \frac{1}{U_2} \right) \right) &\leq 0. \end{aligned}$$

In this here if  $p \in (0, \frac{1}{2})$  then

$$\begin{aligned} 1 - p \left( \frac{1}{l_1} + \frac{1}{U_1} \right) &> 0, \\ 1 - p \left( \frac{1}{l_2} + \frac{1}{U_2} \right) &> 0. \end{aligned}$$

Thus, we get that

$$U_1 - l_1 = 0, U_2 - l_2 = 0.$$

So,  $U_1 = l_1$  and  $U_2 = l_2$ . The proof is completed as desired.  $\square$

## 6. RATE OF CONVERGENCE OF SYSTEM (2.1)

Now we study the rate of convergence of system (2.1). Hence, we consider the following system:

$$(6.1) \quad E_{n+1} = (A + B(n)) E_n,$$

where  $E_n$  is a  $k$ -dimensional vector,  $A \in C^{k \times k}$  is a constant matrix, and  $B : \mathbb{Z}^+ \rightarrow C^{k \times k}$  is a matrix function satisfying

$$(6.2) \quad \|B(n)\| \rightarrow 0,$$

as  $n \rightarrow \infty$ , where  $\|\cdot\|$  denotes any matrix norm that is associated with the vector norm

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$



**Theorem 6.1** (Perron's Theorem, [19]). Assume that condition (6.2) holds. If  $E_n$  is a solution of (6.1), then either  $E_n = 0$  for all  $n \rightarrow \infty$ , or

$$\lim_{n \rightarrow \infty} \sqrt[n]{\|E_n\|},$$

or

$$\lim_{n \rightarrow \infty} \frac{\|E_{n+1}\|}{\|E_n\|},$$

exists and is equal to modulus of one of the eigenvalues of matrix  $A$ .

**Theorem 6.2.** Suppose that  $0 < p < \frac{1}{2}$  and  $\{(t_n, z_n)\}$  be a solution of the system (2.1) such that  $\lim_{n \rightarrow \infty} t_n = \bar{t}$  and  $\lim_{n \rightarrow \infty} z_n = \bar{z}$ . Then the error vector

$$E_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix} = \begin{pmatrix} t_n - \bar{t} \\ t_{n-1} - \bar{t} \\ z_n - \bar{z} \\ z_{n-1} - \bar{z} \end{pmatrix}$$

of every solution of system (2.1) satisfies both of the following asymptotic relations:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{\|E_n\|} &= |\lambda_{1,2,3,4} F_J(\bar{t}, \bar{z})|, \\ \lim_{n \rightarrow \infty} \frac{\|E_{n+1}\|}{\|E_n\|} &= |\lambda_{1,2,3,4} F_J(\bar{t}, \bar{z})|, \end{aligned}$$

where  $\lambda_{1,2,3,4} F_J(\bar{t}, \bar{z})$  are the characteristic roots of the Jacobian matrix  $F_J(\bar{t}, \bar{z})$ .

*Proof.* To find the error terms, we set

$$\begin{aligned} t_{n+1} - \bar{t} &= \sum_{i=0}^1 \alpha_i (t_{n-i} - \bar{t}) + \sum_{i=0}^1 \beta_i (z_{n-i} - \bar{z}), \\ z_{n+1} - \bar{z} &= \sum_{i=0}^1 \gamma_i (t_{n-i} - \bar{t}) + \sum_{i=0}^1 \delta_i (z_{n-i} - \bar{z}), \end{aligned}$$

and  $e_n^1 = t_n - \bar{t}$ ,  $e_n^2 = z_n - \bar{z}$ . Thus we have

$$\begin{aligned} e_{n+1}^1 &= \sum_{i=0}^1 \alpha_i e_{n-i}^1 + \sum_{i=0}^1 \beta_i e_{n-i}^2, \\ e_{n+1}^2 &= \sum_{i=0}^1 \gamma_i e_{n-i}^1 + \sum_{i=0}^1 \delta_i e_{n-i}^2, \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= \alpha_1 = 0, \\ \beta_0 &= \frac{p}{z_{n-1}^2}, \beta_1 = \frac{-p(\bar{z} + z_{n-1})}{\bar{z}z_{n-1}^2}, \\ \gamma_0 &= \frac{p}{t_{n-1}^2}, \gamma_1 = \frac{-p(\bar{t} + t_{n-1})}{\bar{t}t_{n-1}^2}, \\ \delta_0 &= \delta_1 = 0. \end{aligned}$$

Now we take the limits

$$\begin{aligned}\lim_{n \rightarrow \infty} \alpha_0 &= \lim_{n \rightarrow \infty} \alpha_1 = 0, \\ \lim_{n \rightarrow \infty} \beta_0 &= \frac{p}{\bar{z}^2}, \lim_{n \rightarrow \infty} \beta_1 = \frac{-2p}{\bar{z}^2}, \\ \lim_{n \rightarrow \infty} \gamma_0 &= \frac{p}{\bar{t}^2}, \lim_{n \rightarrow \infty} \gamma_1 = \frac{-2p}{\bar{t}^2}, \\ \lim_{n \rightarrow \infty} \delta_0 &= \lim_{n \rightarrow \infty} \delta_1 = 0.\end{aligned}$$

Hence

$$\begin{aligned}\beta_0 &= \frac{p}{\bar{z}^2} + a_n, \beta_1 = \frac{-2p}{\bar{z}^2} + b_n, \\ \gamma_0 &= \frac{p}{\bar{t}^2} + c_n, \gamma_1 = \frac{-2p}{\bar{t}^2} + d_n,\end{aligned}$$

where  $a_n \rightarrow 0, b_n \rightarrow 0, c_n \rightarrow 0, d_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, we obtain the system of the form (6.1)

$$E_{n+1} = (A + B(n)) E_n$$

where

$$\begin{aligned}A &= \begin{pmatrix} 0 & 0 & \frac{p}{\bar{z}^2} & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & 0 & 0 \\ \frac{p}{\bar{t}^2} & \frac{-2p}{\bar{t}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ B(n) &= \begin{pmatrix} 0 & 0 & a_n & b_n \\ 1 & 0 & 0 & 0 \\ c_n & d_n & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\end{aligned}$$

and  $\|B(n)\| \rightarrow 0$  as  $n \rightarrow \infty$ . So, the limiting system of error terms about the equilibrium point  $(\bar{t}, \bar{z})$  can be written as follows:

$$\begin{pmatrix} e_{n+1}^1 \\ e_n^1 \\ e_{n+1}^2 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{p}{\bar{z}^2} & \frac{-2p}{\bar{z}^2} \\ 1 & 0 & 0 & 0 \\ \frac{p}{\bar{t}^2} & \frac{-2p}{\bar{t}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ e_n^2 \\ e_{n-1}^2 \end{pmatrix},$$

which is same as linearized system of system (2.1) about equilibrium point  $(\bar{t}, \bar{z})$ .  $\square$

## 7. NUMERICAL SIMULATIONS

**Example 7.1.** We consider system (2.1) for  $p = 0.43$ . With the initial values  $t_{-1} = 1, t_0 = 1.2, z_{-1} = 3$  and  $z_0 = 0.95$ , positive equilibrium point of system (2.1) is globally asymptotically stable. Figure 1 and Figure 2 verify our theoretical results.

**Example 7.2.** We consider system (2.1) for  $p = 2.2$ . With the initial values  $t_{-1} = 2.08, t_0 = 2.02, z_{-1} = 2.03$  and  $z_0 = 2.08$ , solutions of system (2.1) oscillate about positive equilibrium point  $(\bar{t}, \bar{z}) = (2.0652, 2.0652)$ . Figure 3 verifies our theoretical results.

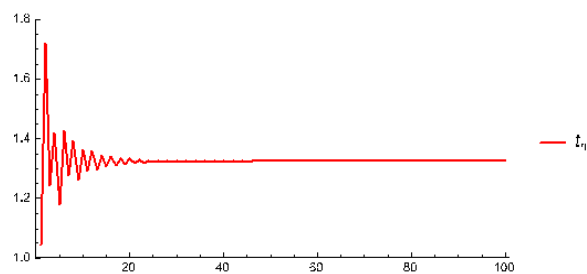
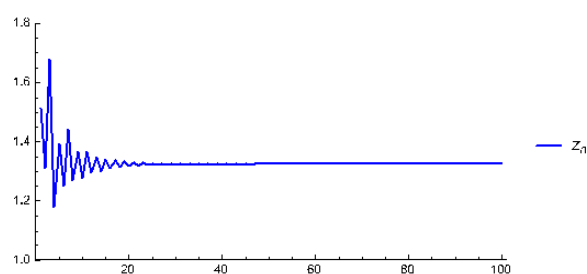
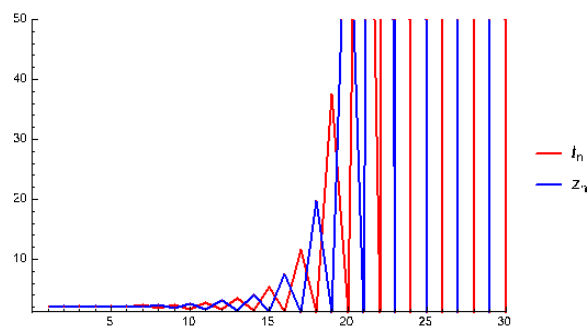
FIGURE 1. Plot of  $t_n$ .FIGURE 2. Plot of  $z_n$ .

FIGURE 3. Plot of system (2.1)

## 8. CONCLUSIONS

This proposed study is related to convergence results of a system of second order difference equations. Firstly we investigate the unique positive equilibrium point of system (2.1). Then we analyse the bounded solutions of system (2.1). We also study the oscillation of solutions of system. Moreover we especially focus on the convergence results of solutions of system. According to our results, if  $0 < p < \frac{1}{2}$  then the positive equilibrium point of system (2.1) is globally asymptotically stable. After this we study the rate of convergence of solutions of system (2.1). In addition to this we present two numerical simulations to verify our theoretical results.

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