

## Analytical solution of D-dimensional radial Schrödinger equation for sextic potential by the extended Nikiforov–Uvarov method and biconfluent Heun polynomials

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**Abstract:** D-dimensional radial Schrödinger equation (SE) for sextic potential is solved using the extended Nikiforov–Uvarov method analytically. Energy eigenvalue and eigenfunction solutions are achieved systematically. It is also presented that the D-dimensional radial SE is transformed to biconfluent Heun equation (BHE). Therefore the eigenfunction solutions for the potential are attained in terms of biconfluent Heun polynomials when the condition of existence of polynomial solution of BHE is provided simultaneously.

**Key words:** D-dimensional Schrödinger equation, extended Nikiforov–Uvarov method, sextic potential, biconfluent Heun polynomials

### 1. Introduction

Solutions of fundamental dynamical equations have received great attention because of its importance in quantum field theory, molecular physics, solid-state physics, and statistical physics. Among the different forms of physical potentials which appear in the wave equations, those with analytical solutions are very restricted. Hence, approximation solution methods or numerical methods are used to obtain their solutions. Since exactly solvable models were still employed extensively as the starting point of numerical methods, analytical analysis of the wave equations have been studied widely [1–9]. In addition to studies of quantum systems in three-dimensional space, scientists need more than  $(3 + 1)$  dimension in order to explain complex systems in quantum mechanics [10–15]. Due to wide-spread applications of the wave equations in D-dimensions in quantum mechanics, exact solutions of these equations have great importance. The traditional solution method for analytic treatment of higher dimensional wave equations is wavefunction ansatz method. However, it includes more details such as numerous coordinate transformations and series expansions. Extended Nikiforov–Uvarov (NU) method, which eliminates the detailed procedure, is developed by changing boundary conditions of the NU method [16]. The NU method is used in order to solve hypergeometric-type second-order differential equations by special orthogonal functions [17]. In general, solution of any second-order differential equations with at most three singular points can be attained exactly using the NU method. By expanding the degrees of polynomial coefficients in the main equation of the NU method, its extended form is derived in order to obtain analytic solution for any second-order differential equations with at most four singular points [16]. The Heun equation and its important confluent forms satisfy the boundary conditions of the extended NU method

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[16, 18, 19]. The Heun equation is a generalized second order differential equation [20, 21]. Studies on analytical solutions of Heun-type equations, which are often encountered in problems in general relativity and astrophysics, have increased extraordinarily in the last two decades (see the bibliography at theheunproject.org) [22]. In the present study, D-dimensional radial SE for sextic potential is solved analytically by the extended NU method. It is presented that the wave equation is reduced to BHE, when the condition for existence of biconfluent Heun polynomials is attained simultaneously. The plan of the paper is as follows: in Section 2, explanation of the extended NU method is given briefly. The exact solution of D-dimensional radial SE for sextic potential has been achieved by the extended NU method in Section 3. Finally, the conclusions are given in the results section.

## 2. The extended NU method

the NU method is a powerful tool in order to achieve exact solution of second-order differential equations which can be reduced to a hypergeometric-type equation given by:

$$\psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(x) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x) = 0, \tag{1}$$

where  $\tilde{\tau}(x)$  is a polynomial of at most first-degree,  $\sigma(x)$  and  $\tilde{\sigma}(x)$  are polynomials of at most second degree. Eq. (1) is the main equation of the NU method. The restriction which is related to degrees of polynomial coefficients in Eq. (1) constitute boundary conditions of the method. Expanding the boundary conditions of the NU method, we proposed a new main equation which is defined as:

$$\psi''(x) + \frac{\tilde{\tau}_e(x)}{\sigma_e(x)}\psi'(x) + \frac{\tilde{\sigma}_e(x)}{\sigma_e^2(x)}\psi(x) = 0, \tag{2}$$

where  $\tilde{\tau}_e(x)$  is a polynomial of at most second degree,  $\sigma_e(x)$  and  $\tilde{\sigma}_e(x)$  are polynomials of at most third and fourth degrees respectively and the subscript  $e$  represents "extended" [16]. Using the transformation:

$$\psi(x) = \phi_e(x)y(x) \tag{3}$$

and the following newly defined polynomials:

$$\tau_e(x) = \tilde{\tau}_e(x) + 2\pi_e(x), \tag{4}$$

$$h(x) = h_n(x) = -\frac{n}{2}\tau'_e(x) - \frac{n(n-1)}{6}\sigma''_e(x) + C_n, \tag{5}$$

$$\pi_e(x) = \frac{\sigma'_e(x) - \tilde{\tau}_e(x)}{2} \pm \sqrt{\left(\frac{\sigma'_e(x) - \tilde{\tau}_e(x)}{2}\right)^2 - \tilde{\sigma}_e(x) + g(x)\sigma_e(x)}, \tag{6}$$

where

$$g(x) = h(x) - \pi'_e(x), \tag{7}$$

the main equation of the extended NU method in Eq. (2) becomes:

$$\sigma_e(x)y''(x) + \tau_e(x)y'(x) + h(x)y(x) = 0. \tag{8}$$

The polynomials  $\tau_e(x)$  with  $\pi_e(x)$  and  $h(x)$  with  $g(x)$ , which are defined in order to rearrange the polynomial coefficients of the main equation, are at most second-degree and first-degree, respectively. To determine all

possible values of  $\pi_e(x)$ , the polynomial  $g(x)$  should be known explicitly. Due to the condition on the degree of  $\pi_e(x)$ , the expression under the square root sign must be square of a polynomial which is at most second-degree. Thus,  $g(x)$  should be specified properly [16]. Each polynomial  $\pi_e(x)$  produces different eigenstate solutions. The eigenvalue solution can be set up by means of Eq. (5). The function  $\phi_e$  is attained from a logarithmic derivative given by:

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)}. \quad (9)$$

Since  $y(x)$  is equal to  $y_n(x)$  which is polynomial solution of Eq. (8), the eigenfunction solution given by the initial transformation  $\psi(x) = \phi_e(x)y(x)$  is achieved completely.

### 3. Solution of D-dimensional radial SE for sextic potential

SE which conserves invariance in spatial rotation for spherically symmetric central fields, is given as for  $\hbar = \mu = 1$  in D-dimensional space:

$$\left[ -\frac{1}{2}\nabla_D^2 + V(r) \right] \psi(\mathbf{r}) = E\psi(\mathbf{r}). \quad (10)$$

The solution of the D-dimensional SE is given as:

$$\psi(\mathbf{r}) = r^{-(D-1)/2} U(r) Y_{l_{D-1} \dots l_1}^l(\hat{\mathbf{x}}), \quad (11)$$

where  $Y_{l_{D-1} \dots l_1}^l(\hat{\mathbf{x}})$  is the generalized spherical harmonics [10]. Substituting this solution into Eq. (10) the radial SE can be obtained as:

$$\left[ \frac{d^2}{dr^2} - \frac{l(l+D-2) + (D-1)(D-3)/4}{r^2} \right] U(r) = -2(E - V(r))U(r). \quad (12)$$

By taking  $\kappa = (l - 1 + D/2)$ , Eq. (12) can be rewritten as [10]:

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa^2 - 1/4}{r^2} \right] U(r) = 2(V(r) - E)U(r). \quad (13)$$

Since the anharmonic oscillator potentials are of great attention for scientists due to their important role in the evolution of many branches in quantum physics, solution of the D-dimensional radial SE for the sextic potential;

$$V(r) = ar^6 + br^4 + cr^2, \quad a > 0 \quad (14)$$

is an interesting physical problem. For this potential, Eq. (13) becomes:

$$\left[ \frac{d^2}{dr^2} - \frac{\kappa^2 - 1/4}{r^2} + 2E - 2ar^6 - 2br^4 - 2cr^2 \right] U(r) = 0. \quad (15)$$

By using the transformation  $(a/2)^{1/4}r^2 = x$ , a comprehensive form of Eq. (15) is achieved as follows:

$$\left[ \frac{d^2}{dx^2} + \frac{1}{2x} \frac{d}{dx} + \frac{-4x^4 - 2b(\frac{2}{a})^{3/4}x^3 - 2c(\frac{2}{a})^{1/2}x^2 + 2E(\frac{2}{a})^{1/4}x - (\kappa^2 - 1/4)}{4x^2} \right] U(x) = 0. \quad (16)$$

Eq. (16) is compared with Eq. (2) and the polynomial coefficients are attained in terms of physical parameters:

$$\begin{aligned}\tilde{\tau}_e(x) &= 1, \\ \sigma_e(x) &= 2x, \\ \tilde{\sigma}_e(x) &= -4x^4 - 2b\left(\frac{2}{a}\right)^{3/4}x^3 - 2c\left(\frac{2}{a}\right)^{1/2}x^2 + 2E\left(\frac{2}{a}\right)^{1/4}x - (\kappa^2 - 1/4).\end{aligned}\tag{17}$$

Then,  $\pi_e(x)$  can be determined by using Eq. (6) for appropriately specified polynomials  $g(x)$ :

$$\pi_{e1}(x) = \frac{1}{2} - 2x^2 - \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}x - \kappa,\tag{18}$$

$$\pi_{e2}(x) = \frac{1}{2} + 2x^2 + \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}x + \kappa\tag{19}$$

for  $g_1(x) = \left[\frac{b^2}{8}\left(\frac{2}{a}\right)^{3/2} + 2\kappa - c\left(\frac{2}{a}\right)^{1/2}\right]x + \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}\kappa + E\left(\frac{2}{a}\right)^{1/4}$ ,  
and

$$\pi_{e3}(x) = \frac{1}{2} + 2x^2 + \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}x - \kappa,\tag{20}$$

$$\pi_{e4}(x) = \frac{1}{2} - 2x^2 - \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}x + \kappa\tag{21}$$

for  $g_2(x) = \left[\frac{b^2}{8}\left(\frac{2}{a}\right)^{3/2} - 2\kappa - c\left(\frac{2}{a}\right)^{1/2}\right]x - \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}\kappa + E\left(\frac{2}{a}\right)^{1/4}$ .

For the polynomial  $\pi_{e1}(x)$ , the analytical solution is given explicitly: The polynomials  $h(x)$  and  $h_n(x)$  are determined by using Eqs. (7) and (5), respectively;

$$h(x) = \left[\frac{b^2}{8}\left(\frac{2}{a}\right)^{3/2} + 2\kappa - c\left(\frac{2}{a}\right)^{1/2} - 4\right]x + \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}(\kappa - 1) + E\left(\frac{2}{a}\right)^{1/4},\tag{22}$$

$$h_n(x) = 4nx + \frac{nb}{2}\left(\frac{2}{a}\right)^{3/4} + C_{n1}.\tag{23}$$

By equating the polynomials  $h(x) = h_n(x)$ , one can reach to the following two equations:

$$\frac{b^2}{2(2a)^{3/2}} + \kappa - \frac{c}{\sqrt{2a}} - 2 = 2n,\tag{24}$$

$$\frac{b}{2}\left(\frac{2}{a}\right)^{3/4}(\kappa - n - 1) + E\left(\frac{2}{a}\right)^{1/4} = C_{n1}.\tag{25}$$

Eq. (24), which is obtained by equating coefficients of  $x$  in Eqs. (22) and (23), implies a constraint on  $\kappa$  and the parameters of the potential. Eq. (25), which is achieved by equating constant terms in Eqs. (22) and (23), gives energy eigenvalue solution when the integration constant  $C_{n1}$  is equal to zero;

$$E = -\frac{b}{\sqrt{2a}}(n - \kappa + 1).\tag{26}$$

For the eigenfunction solution,  $\phi_e$  is identified by Eq. (9) recalling;  $(a/2)^{1/4}r^2 = x$ ;

$$\phi_e(r) = (a/2)^{\frac{1}{8}(-\kappa+\frac{1}{2})}r^{-\kappa+\frac{1}{2}} \exp \left[ -\frac{b}{2\sqrt{2a}}r^2 - \frac{\sqrt{2a}}{4}r^4 \right]. \quad (27)$$

When  $h(x) = h_n(x)$ , Eq. (8) which gives polynomial part of eigenfunction solution is transformed to the following equation:

$$xy''(x) + \left[ 1 - \kappa - \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}x - 2x^2 \right]y'(x) + \left[ 2nx + \frac{b}{4}\left(\frac{2}{a}\right)^{3/4}(\kappa - 1) + \frac{E}{2}\left(\frac{2}{a}\right)^{1/4} \right]y(x) = 0. \quad (28)$$

Eq. (28) is biconfluent form of the Heun equation;

$$xy''(x) + (1 + \alpha - \beta x - 2x^2)y'(x) + \left\{ (\gamma - \alpha - 2)x - \frac{1}{2}[\delta + (1 + \alpha)\beta] \right\}y(x) = 0,$$

where  $\alpha = -\kappa$ ,  $\beta = \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}$ ,  $\gamma = \frac{b^2}{2(2a)^{3/2}} - \frac{c}{\sqrt{2a}}$ , and  $\delta = -E\left(\frac{2}{a}\right)^{1/4}$ . These parameters are achieved by comparing Eqs. (28) and (29) recalling the relation  $\frac{b^2}{2(2a)^{3/2}} + \kappa - \frac{c}{\sqrt{2a}} - 2 = 2n$  given by Eq. (24). Polynomial solution represented with  $N(\alpha, \beta, \gamma, \delta, x)$  of degree  $n$  of the BHE can be achieved in the case of  $\gamma - \alpha - 2 = 2n$  [21]. This relation is directly satisfied in solution processes of the extended NU method. Thus, eigenfunction solution for the potential can be expressed completely:

$$U(r) = (a/2)^{\frac{1}{8}(-\kappa+\frac{1}{2})}r^{-\kappa+\frac{1}{2}} \exp \left[ -\frac{b}{2\sqrt{2a}}r^2 - \frac{\sqrt{2a}}{4}r^4 \right] N \left( -\kappa, \frac{b}{2}\left(\frac{2}{a}\right)^{3/4}, \frac{b^2}{2(2a)^{3/2}} - \frac{c}{\sqrt{2a}}, -E\left(\frac{2}{a}\right)^{1/4}, \left(\frac{a}{2}\right)^{1/4}r^2 \right). \quad (29)$$

By taking the integration constants equal to zero, other eigenstate solutions for the polynomials  $\pi_{e2}(x)$ ,  $\pi_{e3}(x)$ , and  $\pi_{e4}(x)$  can be derived by the above-mentioned procedure;

For Eq. (19),

$$E = -\frac{b}{\sqrt{2a}}(n + \kappa + 1), \quad (30)$$

$$U_2(r) = (a/2)^{\frac{1}{8}(\kappa+\frac{1}{2})}r^{\kappa+\frac{1}{2}} \exp \left[ \frac{b}{4}\left(\frac{2}{a}\right)^{\frac{1}{2}}r^2 + \frac{\sqrt{a}}{2\sqrt{2}}r^4 \right] p_2(r). \quad (31)$$

For Eq. (20),

$$E = \frac{b}{\sqrt{2a}}(\kappa - n - 1), \quad (32)$$

$$U_3(r) = (a/2)^{\frac{1}{8}(-\kappa+\frac{1}{2})}r^{-\kappa+\frac{1}{2}} \exp \left[ \frac{b}{4}\left(\frac{2}{a}\right)^{\frac{1}{2}}r^2 + \frac{\sqrt{a}}{2\sqrt{2}}r^4 \right] p_2(r) \cdot p_3(r). \quad (33)$$

For Eq. (21),

$$E = \frac{b}{\sqrt{2a}}(n + \kappa + 1), \quad (34)$$

$$U_4(r) = (a/2)^{\frac{1}{8}(\kappa+\frac{1}{2})} r^{-\kappa+\frac{1}{2}} \exp \left[ -\frac{b}{2\sqrt{2a}} r^2 - \frac{\sqrt{2a}}{4} r^4 \right]$$

$$N \left( \kappa, \frac{b}{2} \left( \frac{2}{a} \right)^{3/4}, \frac{b^2}{2(2a)^{3/2}} - \frac{c}{\sqrt{2a}}, -E \left( \frac{2}{a} \right)^{1/4}, \left( \frac{a}{2} \right)^{1/4} r^2 \right). \quad (35)$$

#### 4. Results and discussion

Exact eigenstate solutions of D-dimensional radial SE for sextic potential have been achieved using the extended NU method. It is demonstrated that the SE is reduced to BHE. The condition of existence of polynomial solutions of BHE is satisfied simultaneously for the solutions obtained in terms of biconfluent Heun polynomials. Elimination of detailed procedures in power series technique or wavefunction ansatz method is the priority of the method. Since eigenstate solutions are achieved via the extended NU method by the first time for D-dimensional wave equation in an easy way, the extended NU method can be handled as an efficient method for exact solutions of other higher dimensional wave equations for different type potentials.

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